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FINDING SINGLE AND MULTIVARIABLE INTEGRATING FACTORS TO SOME NON-EXACT DIFFERENTIAL EQUATIONS

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ABSTRACT

Finding integrating factors is a method of solving non-exact differential equations. In this paper, the problem of finding integrating factors was stated clearly. Assumptions were made, and solutions were proposed under these assumptions. Specifically three cases were given, and then solutions were given for each case. The first two cases arrived at already existing methods. The third case described a new method for finding an integrating factor, provided some specific conditions are met. The first two cases describe methods for finding single variable integrating factors, whiles the third case describes a method for finding a multivariable integrating factor. In addition, some examples were given for the third case to demonstrate the use of the new method in finding integrating factors.

KEYWORDS:

Non-Exact;
Integrating Factor;
Multivariable;
Differential Equation;
New.

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1. INTRODUCTION

Consider the problem of finding a solution to the differential equation given by

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where the first partial derivatives of $M(x,y)$ and $N(x,y)$ exist. This equation can equivalently be written as

$$M(x,y)dx + N(x.y)dy = 0 \qquad (1)$$

If the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is satisfied, then equation (1) is called "Exact", and (Math24, n.d.)describes a method for dealing with this case. If the condition above is otherwise not satisfied, then equation (1) is called "Non-Exact".

For the non-exact case, we assume the existence of a function, $\mu(x,y)$, called the "Integrating Factor", such that:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x} \qquad (2)$$

If such a function can be found, then the equation

$$\mu M(x,y)dx + \mu N(x,y)dy = 0 \qquad (3)$$

which is exact, can be solved as discussed in(Math24, n.d.).

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2. FINDING THE INTEGRATING FACTOR

Introducing the idea of the integrating factor to solve non-exact differential equations is simple but finding an integrating factor is not as simple. However, there are methods for finding an integrating factor provided certain conditions are met. Some of these methods are discussed in (CliffsNotes, n.d.).

Consider equation (2). Using the product rule of differentiation, and rearranging terms, equation (2) can be written equivalently as

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = M\left(\frac{1}{\mu}\frac{\partial \mu}{\partial y}\right) - N\left(\frac{1}{\mu}\frac{\partial \mu}{\partial x}\right) \qquad (4)$$

Assume the following properties about $\mu(x, y)$:

$$(i)\ \frac{1}{\mu}\frac{\partial \mu}{\partial y} = g(y)$$

$$(ii)\ \frac{1}{\mu}\frac{\partial \mu}{\partial x} = f(x)$$

Then equation (4) is rewritten as

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = Mg(y) - Nf(x) \qquad (5)$$

Note that

$$\mu(x, y) = e^{\int f(x)dx + \int g(y)dy + c} \qquad (6)$$

satisfies the conditions (i) and (ii) above. Consequently, if equation (5) is satisfied, then an integrating factor can be found by equation (6).

We shall examine three cases where the integrating factor can be found.

Case I: f(x) = 0

If f(x) = 0, then equation (5) becomes

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = Mg(y)$$

From this equation, g(y) = $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$, and the integrating factor is found by equation (6). For this case the integrating factor is a function of "y" only.This is the same result as in (CliffsNotes, n.d.), for the case where μ is assumed to be a function of "y"only.

Case II: g(y) = 0

If g(y) = 0, then equation (5) becomes

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -Nf(x)$$

From this equation, f(x) and hence the integrating factor can be found as in Case I. The result found in Case II is also the same as in (CliffsNotes, n.d.), for the case where μ is assumed to be a function of "x" only.

Case III: f(x) and g(y) are non-zero

Define $y_0(x){:}\, M\big(x, y_0(x)\big) = 0$, and $x_0(y){:}\, N(x_0(y), y) = 0$. For $y = y_0(x)$, equation (6) becomes

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)_{y=y_0(x)} = -N\big(x, y_0(x)\big)f(x)$$

Thus $f(x) = \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{-N}\right)_{y=y_0(x)}$ (7).

By a similar approach using $x_0(y)$,

$$g(y) = \left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right)_{x=x_0(y)} \quad (8).$$

An integrating factor can then be found by equation (6). In this case μ is a function of both x and y (i.e. a multivariable integrating factor).

3. SOME EXAMPLES OF CASE III

Examples for Cases I and II can be found in many textbooks e.g.(Zaitsev, 2016)and(Boyce), and also websites such as (CliffsNotes, n.d.) and (Math24, n.d.). Some examples for Case III follow.

(i) $(5xy^2 - 2y)dx + (3x^2y - x)dy = 0$

A quick check will verify that this equation does not satisfy cases I and II. Setting $M(x,y) = 5xy^2 - 2y = 0$ and solving for y gives y = $y_0(x) = \frac{2}{5x}$. Similarly, setting $N(x,y) = 3x^2y - x = 0$ and solving for x gives $x = x_0(y) = \frac{1}{3y}$.

Solving for $f(x)$ as in equation (7) gives $f(x) = \frac{3}{x}$. By a similar approach using equation (8), $g(y) = \frac{1}{y}$.

A quick check shows that forexample (i),$M(x,y)$, $N(x,y)$,$f(x)$ and $g(y)$ satisfy equation (5). Hence an integrating factor is found, using equation (6), to be $\mu(x,y) = Cx^3y,\ for\ all\ real\ C$.

(ii) $\sin(y)\,dx + \sin(x)\,dy = 0$

A quick check here also shows that the above equation does not belong to cases I and II. Setting $M(x,y) = \sin(y) = 0$ and solving for y gives $y = y_0(x) = 0$. Similarly, $x_0(y) = 0$.

Solving for $f(x)$ as in equation (7) gives $f(x) = \frac{1-\cos(x)}{\sin(x)}$. Also, solving for g(y) using equation (8) gives us $g(y) = \frac{1-\cos(y)}{\sin(y)}$. A quick check shows that for this example too,$M(x,y)$, $N(x,y)$, $f(x)$ and $g(y)$ satisfy equation (5). Hence an integrating factor is found, using equation (6), to be $\mu(x,y) = \frac{C}{(1+\cos(x))(1+\cos(y))}$, where C is an arbitrary constant.

4. CONCLUSION

Finding a general expression for an integrating factor for all non-exact differential equations is difficult if not impossible. This paper demonstrated how to find integrating factors for certain non-exact differential equations that satisfy certain conditions. Further work is being done on the method described in Case III to explain the preference of one function over others, when we have multiples zeros of $M(x,y)$ and $N(x,y)$.For instance, in example (i), the functions $y = 0$ and $y = \frac{2}{5x}$ are both possible candidates for $y_0(x)$: $M(x, y_0(x)) = 0$, but only $y = \frac{2}{5x}$ serves our purpose.

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