On the growth analysis of complex linear differential equations with entire and meromorphic coefficients

Sanjib Kumar Datta
BananiDutta

Abstract

The theory of complex differential equation has been developed since 1960’s. Many researchers like Ilpo Laine (1993) have investigated the system of complex differential equation of the following form

\[ f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = F(z) \]

where \( A_i(z) \)'s (\( i=0,1,2,\ldots k-1 \)) and \( F(z) \neq 0 \) are entire or meromorphic functions. The prime concern of this paper is to investigate the comparative growth analysis of the solution as well as the coefficients of the above system of equations.


Keywords:
Entire function, Linear differential equation, Composition, Growth, Entire function of order zero, Meromorphic function, Complex differential equation.

Author correspondence:

1&2 Department of Mathematics, University of Kalyani
P.O.-Kalyani, Dist.-Nadia, PIN-741235.
Email: sanjibdatta05@gmail.com
Email: duttabananimath@gmail.com

1. Introduction

For any two transcendental entire functions \( f \) and \( g \) defined in the open complex plane \( C \), Clunie [3] proved that

\[ \lim_{r \to \infty} \frac{T(r, fog)}{T(r, f)} = \infty \]

and

\[ \lim_{r \to \infty} \frac{T(r, fog)}{T(r, g)} = \infty. \]

Singh [13] proved some comparative growth properties of \( \log T(r, fog) \) and \( T(r, f) \). He also raised the problem of investigating the comparative growth of \( \log T(r, fog) \) and \( T(r, g) \) which he was unable to solve. However some result on comparative growth of \( \log T(r, fog) \) and \( T(r, g) \) are proved later.

Let \( f \) be an entire function defined in the open complex plane \( C \). Known [8] studied on the growth of an entire function \( f \) satisfying second order linear differential equation. Later Chen [4] proved some result on the growth of solutions of second order linear differential equations with meromorphic coefficients. Chen and Yang [5] established a few theorems on the zeros and growths of entire functions of second order linear
The purpose of this paper is to study on the growth of the solution $f \neq 0$ of the $n^{th}$ order linear differential equation 
\[ f^{(n)} + A_1(z)f^{(n-1)} + A_2(z)f^{(n-2)} + \ldots + A_n(z)f = 0, \]
where $A_i(z) \neq 0$ are entire functions. In this paper we investigate the comparative growth of composite entire functions which satisfy $n^{th}$ order linear differential equations.

We do not explain the standard notations and definitions in the theory of entire and meromorphic functions as those are available in [13] and [7].

The following definitions are well known.

**Definition 1** The order $\rho_f$, and lower order $\lambda_f$ of an entire function $f$ is defined as 
\[ \rho_f = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r}, \]
where $\log^{|x|} = \log \left( \log^{|k-1|} x \right)$ for $k = 1, 2, \ldots$ and $\log^{|x|} = x$.

If $f$ is meromorphic, one can easily verify that 
\[ \rho_f = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \lambda_f = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}. \]

**Definition 2** The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of an entire function $f$ is defined as follows 
\[ \overline{\rho}_f = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \liminf_{r \to \infty} \frac{\log M(r,f)}{\log r} \]

If $f$ is meromorphic then 
\[ \overline{\rho}_f = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log r} \quad \text{and} \quad \overline{\lambda}_f = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log r}. \]

**Definition 3** The type $\sigma_f$ of an entire function $f$ is defined as 
\[ \sigma_f = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty. \]

If $f$ is meromorphic then 
\[ \sigma_f = \limsup_{r \to \infty} \frac{\tau(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty. \]

**Definition 4** Let $f$ be an entire function order zero. Then the quantities $\rho_f^+, \lambda_f^+, \overline{\rho}_f^+, \overline{\lambda}_f^+$ are defined in the following way:
\[ \rho_f^+ = \limsup_{r \to \infty} \frac{\log^{|k|} M(r,f)}{\log^{|k|} r} \quad \text{and} \quad \lambda_f^+ = \liminf_{r \to \infty} \frac{\log^{|k|} M(r,f)}{\log^{|k|} r} \]
\[ \overline{\rho}_f^+ = \limsup_{r \to \infty} \frac{\log^{|k|} T(r,f)}{\log^{|k|} r} \quad \text{and} \quad \overline{\lambda}_f^+ = \liminf_{r \to \infty} \frac{\log^{|k|} T(r,f)}{\log^{|k|} r} \]

If $f$ is meromorphic then clearly 
\[ \rho_f^+ = \limsup_{r \to \infty} \frac{\log^{|k|} T(r,f)}{\log^{|k|} r} \quad \text{and} \quad \lambda_f^+ = \liminf_{r \to \infty} \frac{\log^{|k|} T(r,f)}{\log^{|k|} r} \]
\[ \overline{\rho}_f^+ = \limsup_{r \to \infty} \frac{\log^{|k|} T(r,f)}{\log^{|k|} r} \quad \text{and} \quad \overline{\lambda}_f^+ = \liminf_{r \to \infty} \frac{\log^{|k|} T(r,f)}{\log^{|k|} r} \]
**Definition 5** Let ‘a’ be a complex number, finite or infinite. The Nevanlinna deficiency and the Valiron deficiency of ‘a’ w.r.t. a meromorphic function f are defined as

\[
\delta(a,f) = 1 - \lim_{r \to \infty} \sup_{r \neq 0} \frac{N(r,a,f)}{T(r,f)} = \lim_{r \to \infty} \inf_{r \neq 0} \frac{m(r,a,f)}{T(r,f)}
\]

\[
\Delta(a,f) = 1 - \lim_{r \to \infty} \inf_{r \neq 0} \frac{N(r,a,f)}{T(r,f)} = \lim_{r \to \infty} \sup_{r \neq 0} \frac{m(r,a,f)}{T(r,f)}
\]

Now let us define another function:

Let \( \Psi: [0, \infty) \to (0, \infty) \) be a non-decreasing unbounded function, satisfying the following two conditions:

1. \( \lim_{r \to \infty} \frac{\log^{(\alpha)}(\Psi(r))}{\log^{(\alpha)} r} = 0 \)
2. \( \lim_{r \to \infty} \frac{\log^{(\alpha)}(\Psi(r))}{\log^{(\alpha)} r} = 1 \)

for some \( \alpha > 1 \).

With the help of the function \( \Psi \), the classical definitions can be written as,

**Definition 6** The \( \psi \)-order \( \rho_{f,\psi} \) and lower \( \Psi \)-order \( \lambda_{f,\psi} \) of an entire function \( f \) is defined as follows:

\[
\rho_{f,\psi} = \lim_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log \Psi(r)}
\]

\[
\lambda_{f,\psi} = \lim_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log \Psi(r)}
\]

where \( \log^{[k]} x = \log \left( \log^{[k-1]} x \right) \), for \( k = 1, 2, \ldots \), and \( \log^{[0]} x = x \).

If \( \rho_{f,\psi} < \infty \) then \( f \) is of finite \( \psi \)-order. Also, \( \rho_{f,\psi} = 0 \) means that \( f \) is of zero-order. In this connection following Liao and Yang [11] we may give the definition as below:

**Definition 7** Let \( f \) be an entire function of \( \psi \) order zero. Then the quantities \( \rho_{f,\psi}, \lambda_{f,\psi} \) are defined in the following way:

\[
\rho_{f,\psi} = \lim_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log \Psi(r)}
\]

\[
\lambda_{f,\psi} = \lim_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log \Psi(r)}
\]

In the line of Datta and Biswas [6] an alternative definition of zero \( \psi \)-order and zero \( \psi \)-lower order of an entire function may be given as:

**Definition 8** Let \( f \) be an entire function of \( \psi \) order zero. Then the quantities \( \rho_{f,\psi}, \lambda_{f,\psi} \) are defined in the following way:

\[
\rho_{f,\psi} = \lim_{r \to \infty} \frac{\log M(r,f)}{\log \Psi(r)}
\]

\[
\lambda_{f,\psi} = \lim_{r \to \infty} \frac{\log M(r,f)}{\log \Psi(r)}
\]

**Definition 9** The \( \psi \)-type \( \sigma_{f,\psi} \) and \( \psi \)-lower type \( \bar{\sigma}_{f,\psi} \) of an entire function \( f \) are defined as:

\[
\sigma_{f,\psi} = \lim_{r \to \infty} \frac{\log M(r,f)}{\log \Psi(r)}
\]

\[
\bar{\sigma}_{f,\psi} = \lim_{r \to \infty} \frac{\log M(r,f)}{\log \Psi(r)}
\]

\( 0 < \rho_{f,\psi} < \infty \).

2. Research Method: Lemmas

In this section we present some lemmas which will be needed in the sequel.
Lemma 1 [11] If $f$ is meromorphic and $g$ is entire then for all sufficiently large values of $r$,
$$T(r, f) < 1 + o(1)$$
$$\frac{T(r, g)}{\log M(r, g)} T(M(r, g), f)$$

Lemma 2 [2] If $f$ is meromorphic and $g$ is entire and suppose that $0 < \mu \leq \rho \leq \infty$. Then for a sequence of values of $r$ tending to infinity,
$$T(r, f) \geq T(\exp(r^\alpha), f)$$

Lemma 3 [12] If $f, g$ be two transcendental entire functions with $\rho < \infty, \eta$ be a constant satisfying $0 < \eta < 1$ and $\alpha$ be a positive number. Then
$$T(r, f) + O(1) \geq N(r, 0; f) \geq \log \left( \frac{N(M((\eta^r)^{1/2}, r), f)}{\log M((\eta^r)^{1/2}, r) - O(1)} \right)$$
as $r \to \infty$ through all values.

3. Results and Analysis.

In this section we present the main results of the paper.

Theorem 1 Let $f$ be an entire function satisfying the $n$th order linear differential equation
$$f^{(n)} + A_1(z)f^{(n-1)} + A_2(z)f^{(n-2)} + \ldots + A_n(z)f = 0$$
where $A_i(z)$ are non zero entire functions.

(i) $\rho_{A_i} \leq \rho_{A_{i-1}} \leq \ldots \leq \rho_{A_0}$ all are finite,

(ii) $\lambda_{\rho_{A_i}, \rho_{A_{i-1}} \ldots \rho_{A_0}}$ are both non negative,

(iii) $\rho_{A_i} \leq \lambda_{\rho_{A_i}, \rho_{A_{i-1}} \ldots \rho_{A_0}}$ and $\rho_{A_i} \leq \rho_{A_{i-1}} \leq \ldots \leq \rho_{A_0}$ i.e. $\rho_{A_i} \leq \min \lambda_{\rho_{A_i}, \rho_{A_{i-1}} \ldots \rho_{A_0}}$ and $A_i$ be of regular growth, then
$$\lim_{r \to \infty} \frac{\log T(r, (A_1^{\rho_{A_1}} A_2^{\rho_{A_2}} \ldots A_n^{\rho_{A_n}}); A_n)}{T(r, f^{(n)}(A_1^{\rho_{A_1}} A_2^{\rho_{A_2}} \ldots A_n^{\rho_{A_n}}))} = 0.$$
Now combining (1) & (2) it follows for all sufficiently large values of r,

\[
\log T_r(A_1^{\omega} A_2^{\omega - \cdots} A_n) = \frac{\log T_r(A_1^{\omega} A_2^{\omega - \cdots} A_n)}{\mathcal{T}_r(A_1^{\omega} A_2^{\omega - \cdots} A_n)} \leq \limsup_{r \to \infty} \left( e^d(\rho_{A_1^{\omega} A_2^{\omega - \cdots} A_n} + \epsilon) \Psi(r)^{1+\epsilon} \right).
\]

Since, \( \rho_{A_1^{\omega} A_2^{\omega - \cdots} A_n} \leq \lambda_{A_1^{\omega} A_2^{\omega - \cdots} A_n} \), we can choose \( \epsilon > 0 \) in such a way that \( \rho_{A_1^{\omega} A_2^{\omega - \cdots} A_n} + \epsilon < \lambda_{A_1^{\omega} A_2^{\omega - \cdots} A_n} - \epsilon \) and \( \Psi(r) \) is a non-decreasing function, so it follows from above that

\[
\limsup_{r \to \infty} \left( e^d(\rho_{A_1^{\omega} A_2^{\omega - \cdots} A_n} + \epsilon) \Psi(r)^{1+\epsilon} \right) = 0.
\]

Again, for all sufficiently large values of r,

\[
\log T_r(f) \geq [\lambda_{f, \nu} - \epsilon] \log \Psi(r) \hspace{1cm} \text{(4)}
\]

Since, \( \rho_{A_1^{\omega} A_2^{\omega - \cdots} A_n} \leq \lambda_{A_1^{\omega} A_2^{\omega - \cdots} A_n} \), we can choose \( \epsilon > 0 \) in such a way that

\[
\rho_{A_1^{\omega} A_2^{\omega - \cdots} A_n} + \epsilon < \lambda_{A_1^{\omega} A_2^{\omega - \cdots} A_n} - \epsilon. \hspace{1cm} \text{(5)}
\]

Now combining (1), (4) & (5) it follows for all sufficiently large values of r,

\[
\log T_r(A_1^{\omega} A_2^{\omega - \cdots} A_n) \leq \frac{\log T_r(A_1^{\omega} A_2^{\omega - \cdots} A_n)}{\mathcal{T}_r(A_1^{\omega} A_2^{\omega - \cdots} A_n)} \leq e^d(\rho_{A_1^{\omega} A_2^{\omega - \cdots} A_n} + \epsilon) \Psi(r)^{1+\epsilon}.
\]

Therefore in view of (3) and (6), we obtain that

\[
\lim_{r \to \infty} \left( \log T_r(A_1^{\omega} A_2^{\omega - \cdots} A_n) \right) = 0.
\]

**Remark 1** The following example ensures the validity of the conclusion as drawn in Theorem 1.

**Example 1** Let \( n=2 \), \( f = \Psi = \chi_1 \), \( A_1 = \chi_2 \), \( A_2 = \chi_1 \).

Then
\[ A_1 \cdot A_2 = z^2. \]

\[ \log T(r, A_1 \cdot A_2) = \log (2 \log r + O(1)). \]

\[
\rho_{A_1^v} = \lim \sup_{r \to \infty} \frac{\log \log T(r, A_1)}{\log \Psi(z)} = 0, \text{ finite.}
\]

Similarly, \( \lambda_{A_1^v} = 0, \)

\[ \lambda_{A_1^v} = 0. \]

Then

\[
\lim_{r \to \infty} \frac{[\log T(r, A_1 \cdot A_2)]^v}{T(r, f \Psi(r, A_1))} = 0.
\]

**Remark 2** We can choose \( A_i \)'s as meromorphic function for \( i=1,2,\ldots,n-1 \), but \( A_n \) must be an entire function.

Thus in the next example we take \( A_i \) as meromorphic function.

**Example 2** Let \( n=2, f = \Psi = z^2, A_1 = \frac{1}{z-2} \) and \( A_2 = z^2. \)

\[ A_1 \cdot A_2 = \frac{1}{z-2}. \]

\[ \log T(r, A_1 \cdot A_2) = \log (2 \log r + O(1)). \]

\[
\rho_{A_1^v} = \lim \sup_{r \to \infty} \frac{\log \log T(r, A_1)}{\log \Psi(z)} = 0, \text{ finite.}
\]

Similarly, \( \lambda_{A_1^v} = 0, \)

\[ \lambda_{A_1^v} = 0. \]

Thus the conditions are satisfied and

\[
\lim_{r \to \infty} \frac{[\log T(r, A_1 \cdot A_2)]^v}{T(r, F(r, A_1))} = 0.
\]

**Theorem 2** Let \( f \) be an entire function satisfying the \( n \)th order linear differential equation

\[ f^{(n)} + A_1(z)f^{(n-1)} + A_2(z)f^{(n-2)} + \ldots + A_n(z)f = 0, \]

where \( A_i(z) \)'s are non zero entire functions. If \( \lambda_{\{A_1, A_2, \ldots, A_n\}^v} = 0 \) then

\[ \rho_{\{A_1, A_2, \ldots, A_n\}^v} \geq \lambda_{\{A_1, A_2, \ldots, A_n\}^v}^* \mu \]

where \( 0 < \mu < \rho_{\{A_1, A_2, \ldots, A_n\}^v} \).

**Proof:** In view of Lemma 2 and for \( 0 < \mu < \rho_{\{A_1, A_2, \ldots, A_n\}^v} \), we get that

\[ \rho_{\{A_1, A_2, \ldots, A_n\}^v} = \lim \sup_{r \to \infty} \frac{\log T(r, \{A_1, A_2, \ldots, A_n\} \cdot A_1)}{\log \Psi(r)}. \]

\[
\geq \lim \inf_{r \to \infty} \frac{\log T(\exp \Psi(r), \{A_1, A_2, \ldots, A_n\})}{\log \Psi(r)} \]

\[
= \lim \inf_{r \to \infty} \frac{\log T(\exp \Psi(r), \{A_1, A_2, \ldots, A_n\}) \cdot \lim \inf_{r \to \infty} \frac{\log \Psi(r)}{\log \Psi(r)}}{\log \Psi(r)} \]

\[
= \lambda_{\{A_1, A_2, \ldots, A_n\}^v} \cdot \lim \inf_{r \to \infty} \frac{\log (\Psi(r))}{\log \Psi(r)}.
\]
\[
\lambda^* (A_1, A_2, \ldots, A_n) \cdot \mu \cdot \liminf_{r \to \infty} \log \left( \frac{\Psi(r)}{\log M(r, A_n)} \right)
\]

\[
= \lambda^* (A_1, A_2, \ldots, A_n) \cdot \mu.
\]

Thus the theorem is proved.

**Theorem 3** Let \( f \) be an entire function satisfying the \( n \)th order linear differential equation

\[
f^{(n)} + A_1(z)f^{(n-1)} + A_2(z)f^{(n-2)} + \cdots + A_n(z)f = F(z),
\]

where \( A_i(z) \) are non zero entire functions. If

(i) \( \rho_{A_1, A_2, \ldots, A_n} + \rho_{A_i} \) are both finite,

(ii) \( \rho_{A_1, A_2, \ldots, A_n} < \lambda_{r_2, \nu} \) and \( \lambda_{r_2, \nu} \) is positive, then for any \( \alpha \in (-\infty, \infty) \),

\[
\lim_{r \to \infty} \left[ \log \left( T(r, (A_1, A_2, \ldots, A_n)) \right) \right] = 0.
\]

**Proof**: If \( 1 + \alpha \leq 0 \), the theorem is obvious. So we suppose that \( 1 + \alpha > 0 \). In view of Lemma 1, we have for all sufficiently large values of \( r \),

\[
\log \left( T(r, (A_1, A_2, \ldots, A_n)) \right) \log M(r, A_n)
\]

\[
\leq \log T(r, A_n) + \log T(A_1, A_2, \ldots, A_n) + \log \left[ 1 + o(1) \right]
\]

\[
\leq \left( \rho_{A_1, A_2, \ldots, A_n} + \varepsilon \right) \log \Psi(r) + \left( \rho_{A_1, A_2, \ldots, A_n} + \varepsilon \right) \Psi(r) M(r, A_n) + o(1)
\]

\[
\leq \Psi(r)^{\rho_{A_1, A_2, \ldots, A_n} + \varepsilon} \left( \rho_{A_1, A_2, \ldots, A_n} + \varepsilon \right) \log \Psi(r) + o(1) \]

\[
\leq \Psi(r)^{\rho_{A_1, A_2, \ldots, A_n} + \varepsilon} \left( \rho_{A_1, A_2, \ldots, A_n} + \varepsilon \right) \log \Psi(r) + o(1) \]

Again we have for all sufficiently large values of \( r \),

\[
\log T(r, F) \geq \left( \lambda_{r_2, \nu} - \varepsilon \right) \log \Psi(r)
\]

\[
\Rightarrow T(r, F) \geq \Psi(r)^{\lambda_{r_2, \nu} - \varepsilon}
\]

\[
\Rightarrow T(\exp r, F) \geq \Psi(r)^{\lambda_{r_2, \nu} - \varepsilon}, \quad \text{..................(8)}
\]

Now combining (7) and (8), we have for all sufficiently large values of \( r \)

\[
\left[ \log \left( T(r, (A_1, A_2, \ldots, A_n)) \right) \right] \leq \Psi(r)^{\rho_{A_1, A_2, \ldots, A_n} + \varepsilon} \left[ \rho_{A_1, A_2, \ldots, A_n} + \varepsilon \right] \log \Psi(r) + o(1) \]

Since \( \rho_{A_1, A_2, \ldots, A_n} < \lambda_{r_2, \nu} \), we can choose \( \varepsilon > 0 \) in such a way that

\[
\rho_{A_1, A_2, \ldots, A_n} + \varepsilon < \lambda_{r_2, \nu} - \varepsilon
\]
\[
\lim_{r \to \infty} \sup \frac{\log T(r, (A_r A_{r-1} \ldots A_{r-n}) z A_r)}{T(\exp r, F)} \log M(r, A_r)^{\alpha} = 0,
\]

from which the theorem follows.

**Remark 3**: We choose \( f \) instead of \( F \) in the denominator of the statement then the analogous theorem also holds. The following example reveals the fact.

**Example 3**: Let \( n=2 \), \( f = \Psi = z^2 \cdot A_1 = z \cdot A_2 = z^2 \) and \( \alpha = 0 \).

\[
\log T(r, A_1 A_2) = \log(2 \log r + O(1)),
\]

\[
\log M(r, z^2) = 2 \log r,
\]

\[
T(\exp r, z^2) = 2r + O(1).
\]

Then

\[
\lim_{r \to \infty} \frac{\log T(r, A_1 A_2) \log M(r, A_2)}{T(\exp r, F)} = 0.
\]

**Theorem 4**: Let \( f \) be an entire function satisfying the \( n \)th order linear differential equation

\[
f^{(n)} + A_1(z)f^{(n-1)} + A_2(z)f^{(n-2)} + \ldots + A_n(z)f = 0,
\]

where \( A_i(z) \) are non-zero entire functions. If

\[
(i) < \overline{A_i}(A; A_{i-1} \ldots A_{i-n}) A_i < \infty
\]

\[
(ii) < \overline{A_i}(A; A_{i-1} \ldots A_{i-n}) A_i < \infty
\]

Then for any positive number \( \alpha \),

\[
\lim_{r \to \infty} \inf \frac{\log T(r, (A_r A_{r-1} \ldots A_{r-n}) z A_r)}{\log T(\exp r, F)} \leq \overline{\rho}(A; A_{r-1} \ldots A_r; A_r)^{\alpha} \leq \lim_{r \to \infty} \sup \frac{\log T(r, (A_r A_{r-1} \ldots A_{r-n}) z A_r)}{\log T(\exp r, F)} \overline{\rho}(A; A_{r-1} \ldots A_r; A_r)^{\alpha}.
\]

**Proof**: From the definition of hyper-\( \Psi \)-order we get for all sufficiently large values of \( r \),

\[
\log T(r, (A_r A_{r-1} \ldots A_{r-n}) z A_r) \leq \overline{\rho}(A; A_{r-1} \ldots A_r; A_r)^{\alpha} \log \Psi(r)
\]

Again we have for a sequence of values of \( r \) tending to infinity,

\[
\log T(\exp r, F) \leq \overline{\rho}(A; A_{r-1} \ldots A_r; A_r)^{\alpha} \log \Psi(r)
\]

as \( \Psi(r) \) is equivalent to \( \Psi(r^\alpha) \).

Now combining above two equations, it follows for a sequence of values of \( r \) tending to infinity that

\[
\frac{\log T(r, (A_r A_{r-1} \ldots A_{r-n}) z A_r)}{\log T(\exp r, F)} \leq \frac{\overline{\rho}(A; A_{r-1} \ldots A_r; A_r)^{\alpha} \log \Psi(r)}{(\overline{\rho}(A; A_{r-1} \ldots A_r; A_r)^{\alpha} \log \Psi(r))}.
\]

Since \( \epsilon > 0 \) is arbitrary, it follows from above that
\[
\lim \inf_{r \to \infty} \frac{\log^{12} T(r, (A_n^z A_{n-1}^z \ldots A_1^z) A)}{\log^{12} T(f, f)} \leq \frac{\rho(\overline{A_n}, A_n^z, \ldots, A_1^z, A)}{\rho_{f,v}} \] ........(9)

Also for arbitrary positive \( \varepsilon \) and for all sufficiently large values of \( r \),

\[
\log^{12} T(f, f) \geq \left( \frac{\rho_{f,v}}{\rho_{f,v} + \varepsilon} \right) \log \Psi(r) \] ...........(10)

Also for a sequence of values of \( r \) tending to infinity,

\[
\log^{12} T(r, (A_n^z A_{n-1}^z \ldots A_1^z) A) \geq \left( \rho(\overline{A_n}, A_n^z, \ldots, A_1^z, A) - \varepsilon \right) \log \Psi(r) \] ...........(11)

Now from (10) & (11) we obtain for a sequence of values of \( r \) tending to infinity,

\[
\log^{12} T(r, (A_n^z A_{n-1}^z \ldots A_1^z) A) \geq \left( \frac{\rho(\overline{A_n}, A_n^z, \ldots, A_1^z, A) - \varepsilon}{\rho_{f,v} + \varepsilon} \right) \log \Psi(r) \nonumber
\]

Since \( \varepsilon > 0 \) is arbitrary, it follows from above that

\[
\lim \sup_{r \to \infty} \frac{\log^{12} T(r, (A_n^z A_{n-1}^z \ldots A_1^z) A)}{\log^{12} T(f, f)} \geq \left( \frac{\rho(\overline{A_n}, A_n^z, \ldots, A_1^z, A)}{\rho_{f,v}} \right) \] ...........(12)

Then the theorem follows from (9) and (12).

**Remark 4** If \( \psi(z) = z \), we may obtain the corollary.

**Corollary 1** For \( 0 < \rho(\overline{A_n}, A_n^z, \ldots, A_1^z, A) < \infty \) and \( 0 < \rho_{f,v} < \infty \), then for any positive number \( \alpha \),

\[
\lim \inf_{r \to \infty} \frac{\log^{12} T(r, (A_n^z A_{n-1}^z \ldots A_1^z) A)}{\log^{12} T(f, f)} \leq \frac{\rho(\overline{A_n}, A_n^z, \ldots, A_1^z, A)}{\rho_{f,v}} \leq \lim \sup_{r \to \infty} \frac{\log^{12} T(r, (A_n^z A_{n-1}^z \ldots A_1^z) A)}{\log^{12} T(f, f)} \nonumber
\]

**Remark 5** Theorem 4 and Corollary 1 shows that the middle part of the first one is independent of \( \alpha \), where as the second one is dependent on the same.

**Theorem 5** Let \( f \) be an entire function satisfying the \( n^{th} \) order linear differential equation

\[
f^{(n)} + A_n(z) f^{(n-1)} + \ldots + A_1(z) f = 0,
\]

where \( A_i(z) \)s are non zero entire functions. If

\[
(i) \sigma(\overline{A_n}, A_n^z, \ldots, A_1^z, A) < \infty,
\]

\[
(ii) \rho_{f,v} < \infty,
\]

\[
(iii) \sigma_{f,v} < \infty,
\]

\[
(iv) \rho(\overline{A_n}, A_n^z, \ldots, A_1^z, A) = \rho_{f,v}.
\]

Then
\[
\lim \inf_{r \to \infty} \frac{T(r, (A_1 A_2 \cdots A_n)^k A_n)}{T(r, f)} \leq \frac{\sigma(A_1 A_2 \cdots A_n, A_n, r) + \varepsilon}{\sigma_{/\varepsilon}} \leq \lim \sup_{r \to \infty} \frac{T(r, (A_1 A_2 \cdots A_n)^k A_n)}{T(r, f)}.
\]

**Proof** From the definition of \(\psi - \) type, we get for arbitrary positive \(\varepsilon\) and for all sufficiently large values of \(r,\)

\[
T(r, (A_1 A_2 \cdots A_n)^k A_n) \lesssim (\sigma(A_1 A_2 \cdots A_n, A_n, r) + \varepsilon) \psi(r) \rho_{(A_1 A_2 \cdots A_n, A_n, r)}, \quad \ldots \ldots (13)
\]

Again we have for a sequence of values of \(r\) tending to infinity,

\[
T(r, f) \geq (\sigma_{/\varepsilon} - \varepsilon) \psi(r) \rho_{/\varepsilon}, \quad \ldots \ldots (14)
\]

Since \(\rho_{(A_1 A_2 \cdots A_n, A_n, r)} = \rho_{/\varepsilon}\), so from (13) and (14) it follows for a sequence of values of \(r\) tending to infinity,

\[
\lim \inf_{r \to \infty} \frac{T(r, (A_1 A_2 \cdots A_n)^k A_n)}{T(r, f)} \leq \frac{\sigma(A_1 A_2 \cdots A_n, A_n, r) + \varepsilon}{\sigma_{/\varepsilon}} \quad \ldots \ldots (15)
\]

Also for a sequence of values of \(r\) tending to infinity,

\[
T(r, (A_1 A_2 \cdots A_n)^k A_n) \gtrsim (\sigma(A_1 A_2 \cdots A_n, A_n, r) - \varepsilon) \psi(r) \rho_{(A_1 A_2 \cdots A_n, A_n, r)}, \quad \ldots \ldots (16)
\]

Now for all sufficiently large values of \(r,\)

\[
T(r, f) \leq (\sigma_{/\varepsilon} + \varepsilon) \psi(r) \rho_{/\varepsilon}, \quad \ldots \ldots (17)
\]

Now from (16) & (17) we obtain for a sequence of values of \(r\) tending to infinity,

\[
\lim \sup_{r \to \infty} \frac{T(r, (A_1 A_2 \cdots A_n)^k A_n)}{T(r, f)} \gtrsim \frac{\sigma(A_1 A_2 \cdots A_n, A_n, r) - \varepsilon}{\sigma_{/\varepsilon}} \quad \ldots \ldots (18)
\]

Then the theorem follows from (15) and (18).

**Theorem 6** Let \(f\) be a transcendental entire function satisfying the \(n^{th}\) order linear differential equation

\[
f^{(n)} + A_1(z) f^{(n-1)} + A_2(z) f^{(n-2)} + \ldots + A_n(z) f = 0,
\]

where \(A_i(z)\) are non-zero entire functions. If
(i) $0 < \lambda_{A_{n\rightarrow}} \leq \rho_{A_{n\rightarrow}} < \infty$.

(ii) $\lambda_{A_{1^\circ}A_{2^\circ} \cdots A_{n\rightarrow}} > 0$.

(iii) $\rho_{A_{n\rightarrow}} < \infty$.

(iv) $\delta\left(\frac{1}{n} A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\right) < 1$.

Then

$$\limsup_{r \to +\infty} \frac{\log T(r, \{A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\})}{\log \left(\frac{1}{r} f\right)} = \infty,$$

where $\beta$ is a real constant.

**Proof.** We suppose that $\beta > 0$, otherwise the theorem is obvious.

For given $0 < \varepsilon < 1 - \delta\left(\frac{1}{n} A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\right)$,

$$N(r, \{0\}, \{A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\}) = (1 - \delta\left(\frac{1}{n} A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\right) - \varepsilon) T(r, \{A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\}),$$

for a sequence of values or $r$ tending to infinity.

So from Lemma 3, we get for a sequence of values or $r$ tending to infinity,

$$T(r, \{A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\}) = O(1)$$

$$\geq \left[ \log \frac{1}{r} \right] \left[ (1 - \delta\left(\frac{1}{n} A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\right)) - \varepsilon \right] \left[ \frac{1}{\log \left(\frac{1}{r} f\right)} \right] \left[ (A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow})O(1) \right]$$

$$\Rightarrow T(r, \{A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow}\}) = O(1) \geq O(\log r) + O(\log \left(\frac{1}{r} f\right)) \left[ (A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow})O(1) \right]$$

$$+ \log \left[ \frac{\log \left(\frac{1}{r} f\right)}{\log \left(\frac{1}{r} f\right)} \left( A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow} \right) \right]$$

Since $f$ is transcendental, it follows that

$$\lim_{r \to +\infty} \frac{\log \left(\frac{1}{r} f\right)}{\log \left(\frac{1}{r} f\right)} \left( A_{1^\circ} A_{2^\circ} \cdots A_{n\rightarrow} \right) = 0.$$
\[
\frac{\log T'(A_1^2A_2^2\ldots A_n^2),f)}{\log T'(f)} > \frac{O(\log r)}{(1 + \rho A_1A_2\ldots A_n)^r} + \frac{\log T'(A_1^2A_2^2\ldots A_n^2),f)}{\log T'(f)} \\
\limsup_{r \to \infty} \frac{\log T'(A_1^2A_2^2\ldots A_n^2),f)}{\log T'(f)} = \infty.
\]

This proves the theorem.

**Remark 6** If we consider $\Delta$ in the place of $\delta$, then the analogous theorem is also true with ‘limit superior’ replaced by ‘limit’.

**Remark 7** In the theorem using $\Delta$, if we consider $\rho A_1A_2\ldots A_n > 0$ instead of $\rho A_1A_2\ldots A_n > 0$ the theorem remains true with ‘limit’ replaced by ‘limit superior’.

**Conclusion**

The results as deduced in this paper may be thought of from another angle of view and those can be carried out in case of difference polynomials of higher degree. Therefore several modified techniques may be adopted in order to solve the problems arisen and those can be regarded as a virgin area to the researchers in this field.

**References**


********************************************************************************