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Semi-symmetric Metric Connection on a LP-Sasakian Manifold

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Key words :

Semi-symmetric metric connection, LP-Sasakian manifold, Curvature tensor, Ricci tensor.

Abstract

In this paper we have introduced a type of semi-symmetric metric connection on a LP-Sasakian manifold and obtained the expression for curvature tensor. We have also studied conformal curvature tensor, conharmonic curvature tensor, concircular curvature tensor and projective curvature tensor for this connection.

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1. Introduction

Let (V_n, g) be a n-dimensional Riemannian manifold of class C^∞ with metric tensor 'g' and 'D' be Levi-Civita connection on V_n. A linear connection 'E' on (V_n, g) is said to be semi-symmetric ([2]) if the torsion tensor T of the connection 'E' satisfy

$$T(X,Y) = \pi(Y)X - \pi(X)Y, \tag{1.1}$$

where π is a 1-form on V_n with 'ρ' as associated vector field, i.e.

$$\pi(X) = g(X,\rho), \tag{1.2}$$

for arbitrary vector field X on V_n.

A semi-symmetric connection 'E' is called semi-symmetric metric connection ([4]) if it further satisfies

$$E_X g = 0. \tag{1.3}$$

Let V_n is a 'n' dimension C^∞ manifold. On V_n there exist a tensor 'F' of type (1, 1), a vector field U, a 1-form 'u' and Lorentzian metric 'g' such that

$$\bar{\bar{X}} = X + u(X)U, \tag{1.4}$$

$$u(\bar{X}) = 0, \tag{1.5}$$

$$g(\bar{X}, \bar{Y}) = g(X,Y) + u(X).u(Y), \tag{1.6}$$

$$g(X,U) = u(X), \tag{1.7}$$

$$(D_X F)Y = g(X,Y).U + u(Y).X + 2u(X).u(Y)U, \tag{1.8}$$

$$D_X U = \bar{X}, \tag{1.9}$$

where $F(X) \overset{\text{def.}}{=} \bar{X}$ for arbitrary vector field X, Y. Then V_n satisfying above equations is called LP-Sasakian manifold and $\{F, u, U, g\}$ is called LP-Sasakian structure on V_n. Here 'D' is Levi-Civita connection with respect to 'g'.

In LP-Sasakian manifold, we have

$$u(U) = -1, \tag{1.10}$$

$$\text{rank}\,(F) = n-1, \tag{1.11}$$

$$g(\bar{X}, Y) = g(X, \bar{Y}). \tag{1.12}$$

Let us define fundamental 2-form $'F$ on a LP-Sasakian manifold as below

$$'F(X,Y) = g(\bar{X}, Y), \tag{1.13}$$

then, we have

$$'F(X,Y) = {}'F(Y,X), \tag{1.14}$$

$$'F(\bar{X}, \bar{Y}) = {}'F(X,Y), \tag{1.15}$$

and

$$'F(X,Y) = (D_X u)Y, \tag{1.16}$$

On a LP-Sasakian manifold, we can easily verify

$$(D_X{}'F)(Y,Z) = g(X,Y).u(Z) + g(X,Z).u(Y) + 2u(X).u(Y).u(Z), \tag{1.17}$$

$$(D_X{}'F)(Y,U) = -g(\bar{X},\bar{Y}), \tag{1.18}$$

$$g(K(X,Y,Z),U) = u(K(X,Y,Z)) = g(Y,Z).u(X) - g(X,Z).u(Y), \tag{1.19}$$

$$K(X,Y,U) = u(Y)X - u(X)Y, \tag{1.20}$$

$$K(U,X,Y) = g(X,Y).U - u(Y)X, \tag{1.21}$$

$$K(U,X,U) = X - u(X).U, \tag{1.22}$$

$$\text{Ric}(X,U) = (n-1)u(X). \tag{1.23}$$

2. Semi-symmetric metric connection

On V_n, we define a connection 'E' satisfying

$$E_X Y = D_X Y + u(Y).X - g(X,Y)U \tag{2.1}$$

$$E_X g = 0, \tag{2.2}$$

then torsion 'T' of 'E' is given by

$$T(X,Y) = E_X Y - E_Y X - [X,Y] = u(Y)X - u(X)Y. \tag{2.3}$$

Here E is called semi-symmetric metric connection on LP-Sasakian manifold V_n.

Let 'R' is curvature tensor of E and 'K' is curvature tensor of connection D, then

$$R(X,Y,Z) = K(X,Y,Z) + \{g(\bar{Y},\bar{Z}) - g(\bar{Y},Z)\}X - \{g(\bar{X},\bar{Z}) - g(\bar{X},Z)\}Y$$

$$+u(X).g(Y,Z).U - u(Y).g(X,Z).U + g(X,Z)\bar{Y} - g(Y,Z)\bar{X}, \tag{2.4}$$

where

$$K(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z.$$

Contracting (2.4) with respect to X

$$\tilde{\text{Ric}}(Y,Z) = \text{Ric}(Y,Z) + (n-1)\{g(\bar{Y},\bar{Z}) - g(\bar{Y},Z)\}$$

$$-g(Y,Z) - u(Y).u(Z) + g(\bar{Y},Z) \tag{2.5}$$

i.e. $$\tilde{\text{Ric}}(Y,Z) = \text{Ric}(Y,Z) + (n-2)\{g(\bar{Y},\bar{Z}) - g(\bar{Y},Z)\}.$$

Contracting (2.5), we get

$$\tilde{R}Y = RY + (n-1)\{\bar{\bar{Y}} - \bar{Y}\} - Y - u(Y).U + \bar{Y}$$

i.e.

$$\tilde{R}Y = RY + (n-2)\{\bar{\bar{Y}} - \bar{Y}\}, \qquad (2.6)$$

i.e.

$$\tilde{R}Y = (n-1)Y + (n-2)\{Y + u(Y).U - \bar{Y}\}. \qquad (2.7)$$

Contracting above equation with respect to '*Y*'

$$\tilde{r} = r + (n-1)(n-2), \qquad (2.8)$$

where $\tilde{r}$ and '*r*' are scalar curvature with respect to *E* and *D* in V_n.

3. Properties of Semi-symmetric Metric Connection

Theorem 3.1. In V_n, we have

(i) $(E_X F)Y = \{g(X,Y) - g(X,\bar{Y})\}.U + u(Y)(X - \bar{X}) + 2u(X).u(Y).U$ (3.1)

(ii) $E_X U = \bar{X} - \bar{\bar{X}}$ (3.2)

(iii) $(E_X u)Y = g(X,\bar{Y}) - g(\bar{X},\bar{Y})$ (3.3)

Proof. As we know that

(i) $(E_X F)Y = E_X \bar{Y} - \overline{E_X Y}$

$$= D_X\bar{Y} + 0 - g(X,\bar{Y})U - \overline{D_X Y} - u(Y)\bar{X} + 0 = (D_X F)Y - g(X,\bar{Y}).U - u(Y)\bar{X}$$

$$= g(X,Y).U + u(Y)X + 2.u(X).u(Y).U - g(X,\bar{Y}).U - u(Y)\bar{X}$$

$$= \{g(X,Y) - g(X,\bar{Y})\}.U + u(Y)(X - \bar{X}) + 2u(X).u(Y).U.$$

From (2.1), we have

(ii) $E_X U = D_X U + u(U).X - g(X,U)U$

$$= \bar{X} - X - u(X).U = \bar{X} - \bar{\bar{X}}.$$

(iii) Taking covariant derivative of $u(Y)$ with respect to connection '*E*' and '*D*', we get

$$X(u(Y)) = (E_X u)Y + u(E_X Y)$$

and

$$X(u(Y)) = (D_X u)Y + u(D_X Y)$$

from above two equations, we have

$$0 = (E_X u)Y - (D_X u)Y + u(E_X Y - D_X Y)$$

$$(E_X u)Y = (D_X u)Y - u(X).u(Y) - g(X,Y) = g(X,\bar{Y}) - g(\bar{X},\bar{Y}).$$

Theorem 3.2. In V_n, we have

$$R(X,Y,Z)+R(Y,Z,X)+R(Z,X,Y)=0. \tag{3.4}$$

Proof. By cyclic rotation of *X*, *Y*, *Z* in equation (2.4), we get three equations. Adding these three equations and using

$$K(X,Y,Z)+K(Y,Z,X)+K(Z,X,Y)=0.$$

We get the required result.

Theorem 3.3. In V_n, the conformal curvature tensor $\tilde{Q}$ with respect to semi-symmetric metric connection *E* is same the conformal curvature tensor with respect to Levi-Civita connection *D*.

Proof. Conformal curvature tensor with respect to connection '*E*' and '*D*' denoted by $\tilde{Q}$ and *Q* are defined as

$$\tilde{Q}(X,Y,Z)=R(X,Y,Z)-\frac{1}{n-2}\{\widetilde{\text{Ric}}(Y,Z)X-\widetilde{\text{Ric}}(X,Z)Y+g(Y,Z)\tilde{R}X$$

$$-g(X,Z)\tilde{R}Y\}+\frac{\tilde{r}}{(n-1)(n-2)}\{g(Y,Z)X-g(X,Z)Y\}, \tag{3.5}$$

$$Q(X,Y,Z)=K(X,Y,Z)-\frac{1}{n-2}\{\text{Ric}(Y,Z)X-\text{Ric}(X,Z)Y+g(Y,Z)RX$$

$$-g(X,Z)RY\}+\frac{r}{(n-1)}\{g(Y,Z)X-g(X,Z)Y\}, \tag{3.6}$$

where $R(X,Y,Z)$ and $K(X,Y,Z)$, $\widetilde{\text{Ric}}$ and Ric, $\tilde{R}$ and *R*, $\tilde{r}$ and *r* are curvature tensor, Ricci tensor, Ricci map and scalar curvature with respect to connection *E* and *D*, respectively.

Using (2.4), (2.5), (2.6), (2.7) and (2.8) in equation (3.5), we get

$$\tilde{Q}(X,Y,Z)=Q(X,Y,Z). \tag{3.7}$$

Corollary 3.3. If V_n is a LP-Sasakian space form then it is conformally flat with respect to *E*.

Proof. If V_n is a LP-Sasakian space form, then we have

$$K(X,Y,Z)=g(Y,Z)X-g(X,Z)Y.$$

In this manifold, we have

$$Q(X,Y,Z)=0.$$

Using this fact in (3.7), we get the result.

Theorem 3.4. In V_n, the conharmonic curvature tensor $\tilde{L}$ with respect to connection E is given as

$$\tilde{L}(X,Y,Z) = L(X,Y,Z) - \{g(Y,Z)X - g(X,Z)Y\} \tag{3.8}$$

where 'L' is conharmonic curvature tensor with respect to connection D.

Proof. As we know that conharmonic curvature tensor $\tilde{L}$ with respect to semi-symmetric metric connection 'E' is given as

$$\tilde{L}(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-2}\{\tilde{R}ic(Y,Z)X - \tilde{R}ic(X,Z)Y + g(Y,Z)\tilde{R}X - g(X,Z)\tilde{R}Y\}. \tag{3.9}$$

Using (2.4), (2.5), (2.6) and (2.7) in (3.9), we get the required result.

Corollary 3.4. If V_n is a LP-Sasakian space form, then we have

$$\tilde{L}(X,Y,Z) = -\frac{2(n-1)}{(n-2)}K(X,Y,Z). \tag{3.10}$$

Proof. If V_n is a LP-Sasakian space form, then we have

$$K(X,Y,Z) = g(Y,Z)X - g(X,Z)Y \tag{3.11}$$

$$L(X,Y,Z) = \frac{n}{n-2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{3.12}$$

Using (3.11) and (3.12) in (3.8), we get the required result.

Theorem 3.5. The concircular curvature tensor $\tilde{C}$ with respect to semi-symmetric metric connection 'E' is given by in V_n, the conharmonic curvature tensor $\tilde{L}$ with respect to connection E is given as

$$\tilde{C}(X,Y,Z) = C(X,Y,Z) + \{g(\bar{Y},\bar{Z}) - g(\bar{Y},\bar{Z})\}X - \{g(\bar{X},\bar{Z}) - g(\bar{X},Z)\}Y - g(Y,Z)\bar{X} + g(X,Z)\bar{Y}$$

$$+g(Y,Z)u(X)U - g(X,Z)u(Y).U - \frac{(n-2)}{n}\{g(Y,Z)X - g(X,Z)Y\} \tag{3.13}$$

Proof. As we know that conharmonic curvature tensor $\tilde{C}$ with respect to E is define as

$$\tilde{C}(X,Y,Z) = R(X,Y,Z) - \frac{\tilde{r}}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{3.14}$$

Putting the value of $R(X,Y,Z)$ and $\tilde{r}$ from equation (2.4) and (2.8) in (3.14), we get the required result.

Corollary 3.5. If V_n is a LP-Sasakian space form, then concircular curvature tensor $\tilde{C}$ with respect to E can also be given by equation

$$\tilde{C}(X,Y,Z) = \frac{2}{n}K(X,Y,Z) + u(Z).K(X,Y,U) - K(\bar{X},Y,Z)$$

$$-K(X,\bar{Y},Z) + u(K(X,Y,Z)).U. \tag{3.15}$$

Proof. In LP-Sasakian space form V_n, we know that

$$K(X,Y,Z) = g(Y,Z)X - g(X,Z)Y,$$

$$K(\bar{X},Y,Z) = g(Y,Z)\bar{X} - g(\bar{X},Z)Y,$$

$$K(X,\bar{Y},Z) = g(\bar{Y},Z)X - g(X,Z)\bar{Y},$$

$$K(X,Y,U) = u(Y).X - u(X)Y,$$

$$u(K(X,Y,Z)) = g(Y,Z)u(X) - g(X,Z)u(Y),$$

$$g(\bar{X},\bar{Y}) = g(X,Y) + u(X)u(Y),$$

$$C(X,Y,Z) = 0.$$

Using these results in (3.13), we can easily get (3.15).

Theorem 3.6. The projective curvature tensor $\tilde{P}$ in V_n with respect to connection E is given by

$$\tilde{P}(X,Y,Z) = P(X,Y,Z) + \frac{1}{n-1}[\{g(\bar{Y},\bar{Z}) - g(\bar{Y},Z)\}X + \{g(\bar{X},\bar{Z}) - g(\bar{X},Z)\}Y] - g(Y,Z)\bar{X}$$

$$+g(X,Z)\bar{Y} + g(Y,Z)u(X)U - g(X,Z)u(Y).U. \qquad (3.16)$$

Proof. Projective curvature tensor $\tilde{P}$ with respect to connection E is given by

$$\tilde{P}(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[\tilde{R}\mathrm{ic}(Y,Z)X - \tilde{R}\mathrm{ic}(X,Z)Y]. \qquad (3.17)$$

Using (2.4) and (2.5) in above, we get the required result.

References

1. **De, U. C. :** On type of semi - symmetric metric connection on a Riemannian manifolds, An. Stiint. Univ. "Al I. Cuza" Iasi Sect. I Math., 38 (1991), 105 - 108.

2. **Fridmann, A. and Schouten, J. A. :** Über die Geometric der holbsymmetrischen, Übertragurgen Math. Z 21 (1924), 211-233.

3. **Mishra, R. S. and Pandey, S. N. :** On quarter-symmetric metric F connections, Tensor, N. S., 34 (1980), 1-7.

4. **Sharfuddin, A. and Hussain, S. I. :** Semi-symmetric metric connection in almost contact manifold.