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A Brief Study of Group Theory

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Group:-

The term group was coined by galois around 1830 to describe sets of one - to - one functions on finite sets that could be grouped together to form a closed set. As in the case with most fundamental concepts in mathematics, the modern definition of a group that follows is the result of a long evolutionary process.

Definition:-

A non – empty set G along with a binary operation '*' is said to be a group i.e., (G, *) is a group if it satisfy the following four properties:-

1. Closure property :- The set G is closed with respect to the composition ' * ' ,i.e. $a, b \in G \Rightarrow a * b \in G$
2. Associative law :- The binary operation obeys associative law i.e. $(a * b) * c = a * (b * c) \; \forall \; a, b, c \in G$
3. Existence of identity element :- For all $a \in$ G there exists an element e ∈ G such that $a * e = a = e * a$

   The element e is called the identity element of G.
4. Existence of Inverse: - For each $a \in$ G there exists a unique element $b \in$ G s.t. a*b = b*a = e

   The element b is called inverse of a.

Subgroup :-

In group theory, a branch of mathematics, given a group G under a binary operation *, a subset H of G is called a subgroup of G if H also forms a group under the operation *.

The trivial subgroup of any group is the subgroup {e} consisting of just the identity element.

A proper subgroup of a group G is a subgroup H which is proper subset of H. i.e. $H \neq G$.

If H is a subgroup of G, the G is sometimes called an overgroup of H.

Basic properties of subgroup of subgroups:-

1. A subset H of a group G is a subgroup of G if and only if it is nonempty and closed under products and inverse. In case that H is finite, then H is a subgroup if and only if H is closed under products.
2. H is a subgroup of G iff H is a subset of G and there is an inclusion homomorphism.
3. The identity of a subgroup is the identity of group: if G is a group with identity e_G and H is subgroup of G with identitye_H then $e_H = e_G$eH.
4. The inverse of an element in a subgroup is the inverse of the element in the group, if H is a subgroup of group G and a & b are elements of H s.t.$ab = ba = e_H$then a$b = ba = e_G$.
5. The intersection of subgroups A and B is again a subgroup. The union of subgroups A and B is a subgroup if and only if either A or B contains the other.

For example $H =$ 2H = 2Z is a subgroup of Z &

K = 5Z is a subgroup of Z s.t.

HUK = 2Z U 5Z is not a subgroup of Z

Because, 2∈2Z U 5Z & 5∈2Z U 5Z

But 2+5 = 7 ∉2Z U 5Z

Coset:-

Let G be a group and H is any subgroup of G. For any a∈G , the set Ha = { ha; h∈ H } is called right cooet of H in G generated by a.

Similarly; the set aH = {ah; h∈H} is called left cooet of H in G generated by a.

Ha &aH are both Subsets of G. H is itself a right and left coset as eH=H=He , where e is identify of G. Since H is a Subgroup of G, therefore $e \in H \Longrightarrow ea \in Ha$& hence right coset is a non-empty set. Also $e \leftarrow H \Rightarrow ae \in aH$& hence left coset is also a non-empty set.If G is an abelian group and H is Subgroup of G, then ah=ha for all $h \in H$.

Hence aH=Ha.

Normal subgroup:-

A subgroup of a group G is called a normal subgroup if it is invariant under conjugation of an element of N by an element of G is still in N.

i.e. $\forall n \in N, \forall g \in G, gng^{-1} \in N$.

The following conditions are equivalent to normality:-

- Any two elements commute regarding the normal subgroup membership relation:

$\forall g, h \in G, gh \in N \Leftrightarrow \in N$

- The image of conjugation of N by any element of G is subset of N.$\forall g \in G, gNg^{-1} \subseteq N.$
- The image of conjugation of N by any element of G is N.$\forall g \in G, gNg^{-1} = N.$
- The sets of left & right cosets of N in G concide: c$\forall g \in G, gN = Ng.$

For example

1. The subgroup {e} consisting of just the identity element of G is normal subgroup of G
2. The group G itself is normal Subgroup of G and these called trivial Subgroups.
3. The Center of group is a normal subgroup.
4. All subgroups of an abelian group are normal.

Properties of normal subgroup:-

1. Normality is preserved upon surjective homomorphism and is also preserved upon taking inverse image.
2. Normality is preserved on taking direct products.
3. If $H \subseteq K \subseteq G$; where H & K are subgroups of G and His normal subgroup of K.
4. If $H \subseteq K \subseteq G$& H is normal subgroup of K, K is normal subgroup of G then H need not be normal subgroup of G.
5. If H is subgroup of G with index 2, then H is always normal subgroup of G.

Index of Subgroup:-

Let H be subgroup of G. The number of cosets (left coset or right coset) of H in G is called index of subgroup H in G.

Centralizer of a subset S of a group G :-

C_G(S) ={$g \in G; gs = sg \ for \ all \ s \in S$}

When S= {a} is a singleton set, then C_G\{(a)\} can be abbreviated to C_G(a). or Z(a).

Normalizer of S in the group G:-

N_G(S)={ $g \in G; gS = Sg$}

The definitions are similar but not identical. If g is in the centralizer of S and s is in S, then it must be that gs=sg , however if g is in the normalize , gs=tg for some t in S may be different from s.

Simple Group:-

If a group G has no proper normal subgroups, that is,if the only normal subgroups of G are {e} & G itself , then G is called simple group. If the order of group is finite then G is called finite simple group.

Theorem:-

Every finite simple group is isomorphic to one of the following groups:

- A member of one of three infinite classes of such namely
  - The cyclic groups of prime orders.
  - The alternating groups of degree atleast 5
  - The group of Lie type
- One of 26 groups called the 'Sporadic groups'
- The Tits group.

Sylow theorems:-

A group G is said to be p-Group if $O(G) = p^n$.

Sylow First theorem:-

If G is finite group and $p^n/O(G)$then G has subgroup of order p^n.

Sylow p-Subgroup or p-SSG:-

If G is finite group &$p^n/O(G)$. But $p^{n+1}Xo(G)$ then the Subgroup of order p^n is called p-Subgroup of p-SSG.

Sylow Second theorem:-

Any two p-SSG of G are Conjugate, Where G is a finite group. i.e.if H & K are P-SSG of G, Then there exists an element x in G with $x^{-1}Hx = K$.

Sylow's Third Theorem:-

If G is finite group and p is a prime number, then the number of p-SSG(n_p) is equal to1+kp s.t.

1+kp/0(G); K=0,1,2,….i-e;

n_p=1+pk s.t. 1+pk/0(G), k=0,1,2,….

Types of group:-

- Simple group
- Finite group
- Abelian group

    - Torsion subgroup
    - Free abelian group
    - Finitely generated abelian group
- Cyclic group
- Solvable group
- Nilpotent group
- Hamiltonian group

Example of groups:-

- Permutation group
- Symmetric group
- Alternating group
- P-group
- Klein four- group
- Quaternion-group
- Dihedral group
- Dicyclic group
- Auto morphism group