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A NOTE ON THE LAPLACE TRANSFORM OF THE GENERALIZED INCOMPLETE GAMMA FUNCTION

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Abstract

A few days back, while working on the Extended Beta and Gamma functions of matrix arguments [1], the present author has found a relation between the generalized incomplete gamma function of Chaudhry and Zubair [2] and one of the Humbert's confluent hypergeometric functions of two variables. We show in the present note that the Laplace transform of the generalized incomplete gamma function of Chaudhry and Zubair [2] can be evaluated in terms of this Humbert's function. The result which we establish in this paper for the special functions concerned with scalar arguments will be generalized to the case of matrix arguments in a further communication of this author in the near future.

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1. Introduction

While teaching the chapters on permutation and combination at the secondary school level in mathematics courses we introduce the concept of the 'factorial' for the first time to the students. Later on, in the undergraduate courses in mathematics, especially while teaching calculus of a single variable in the real domain i.e. calculus of functions of a single real variable or in the courses dealing with the theory of complex variable or advanced calculus [3] the student is usually taught about the gamma function . Needless to mention here that the domain of the factorial function was extended from the set of natural numbers to the set of real numbers (more accurately to say, the set of complex numbers) by Euler (1707-1783). Though, there are many ways of defining the gamma function, here we define it as an improper integral, as is very widely available in the literature (see, for instance, Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. Vol I [4, (1), p.1], Whittaker and Watson [5, Chapter XII, p. 235], Lebedev [6, (1.1.1), p.1], just to mention a few)

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$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt \qquad \mathrm{Re}(z)>0, \tag{1.1}$$

where $\mathrm{Re}(z)$ denotes the real part of the complex variable (z). It may be of interest to mention here that the notation $\Gamma(z)$ for the gamma function and the name 'gamma function' itself were introduced by Legendre in 1814 (see, for instance, was Whittaker and Watson [5, Chapter XII, p. 235] or Chaudhry and Zubair [2, p.1, Chapter 1].

The above stated classical gamma function can be extended in a number of ways. One such useful extension of the said classical gamma function defined by (1.1) above is due to Chaudhry and Zubair [2, (1.66), p.9] which, in terms of their notation [2, (1.66), p.9] is defined as below

$$\Gamma_b(\alpha)=\int_0^\infty e^{-t-b/t}t^{\alpha-1}dt \tag{1.2}$$

for $(\mathrm{Re}(b)>0; b=0, \mathrm{Re}(\alpha)>0)$.

According to Chaudhry and Zubair [2], the introduction of the factor $e^{-b/t}$ in the integrand of (1.1) plays the role of the regularizer [2, p. 9] (see also Chaudhry, Qadir, Rafique, Zubair [7, p. 20]). It may further be noted from (1.2) above that for $\mathrm{Re}(b)>0$, the function $\Gamma_b(\alpha)$ is defined throughout the complex plane and for $b=0$, it reduces to the classical gamma function of (1.1) (see Chaudhry and Zubair [2 , p. 9]).

The incomplete gamma function $\gamma(z,\alpha)$ and its complementary function $\Gamma(z,\alpha)$ (also called the complementary incomplete gamma function, see Luke [8, p. 342] are defined respectively by the following equations

$$\gamma(z,\alpha)=\int_0^\alpha e^{-t}t^{z-1}dt, \qquad \mathrm{Re}(z)>0, |\arg\alpha|<\pi \tag{1.3}$$

$$\Gamma(z,\alpha)=\int_\alpha^\infty e^{-t}t^{z-1}dt, \qquad |\arg\alpha|<\pi \tag{1.4}$$

(see, for instance, Lebedev [8, exercise 10, p. 15], Luke [8, , 6.2.11 (1) and (2), p. 220-221], Whittaker and Watson [5, p. 341], Andrews, Askey and Roy [9, (4.4.5), p. 197], Chaudhry and Zubair [2, (2.1) and (2.2) , p. 37]).

Chaudhry and Zubair [2, (2.64), (2.65), p. 43] have defined the generalized incomplete gamma function and its complement by the following equations:

$$\gamma(\alpha,x;b)=\int_0^x e^{-t-bt^{-1}}t^{\alpha-1}dt \tag{1.5}$$

$$\Gamma(\alpha,x;b)=\int_x^\infty e^{-t-bt^{-1}}t^{\alpha-1}dt \tag{1.6}$$

where, in (1.5) and (1.6) α, x are complex parameters and b is a complex variable. For the special case, $b=0$, (1.5) and (1.6) reduce respectively to (1.3) and (1.4) (subject, of course, to suitable modifications) i.e., to say,

$$\gamma(\alpha,x;0)=\gamma(\alpha,x) \tag{1.7}$$

$$\Gamma(\alpha,x;0)=\Gamma(\alpha,x) \tag{1.8}$$

In this note we propose to work with the generalized incomplete gamma function of Chaudhry and Zubair defined by (1.5) above. The scheme of the paper is as follows - in section 2 the basic concepts and definitions of Laplace transform of a function, some known results, which exist in the literature, are given which shall be used by us in the sequel. In section 3 of the paper the main result will be stated in the form of a theorem and it will be proved with the results and definitions enumerated in sections 1 and 2.

2. Preliminary Results and Definitions

Definition 2.1: The Laplace transform of a function $F(t)$, denoted by $L\{F(t);s\}$, is defined as

$$L\{F(t);s\} = \int_0^{\infty} e^{-st} F(t)\,dt \qquad (2.1)$$

(e.g., see Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. Vol I [4, 6.10(1), p. 269]).

Definition 2.2: The incomplete beta function, represented by $B_x(\nu,\mu)$, is defined by the following integral relation

$$B_x(\nu,\mu) = \int_0^x t^{\nu-1}(1-t)^{\mu-1}\,dt, \qquad (0 \le x \le 1) \qquad (2.2)$$

(see Chaudhry and Zubair [2, (5.25), p. 217]. It can be shown that the incomplete beta function $B_x(\nu,\mu)$ satisfies the relations

$$B_x(\nu,\mu) = x^{\nu}\int_0^1 t^{\nu-1}(1-xt)^{\mu-1}\,dt \qquad (2.3)$$

(Chaudhry and Zubair [2, (5.48), p. 219]) and

$$B_x(\nu,\mu) = \frac{x^{\nu}}{\nu} \times {}_2F_1(\nu, 1-\mu; 1+\nu; x) \qquad (2.4)$$

(Chaudhry and Zubair [2, (5.27), p. 217]). In (2.4), ${}_2F_1$ represents the celebrated Gauss hypergeometric function, defined by the equation

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \qquad (2.5)$$

(see, for instance, Srivastava and Karlsson [10, (17), p.18] whose convergence condition requires that $|z|<1$ and $c \ne 0,-1,-2,\cdots$. When $|z|=1$ then it can be shown that the series in (2.5) is

(i) absolutely convergent if $\mathrm{Re}(c-a-b) > 0;$

(ii) conditionally convergent if $-1 < \mathrm{Re}(c-a-b) \le 0, z \ne 1;$

(iii) divergent if $\mathrm{Re}(c-a-b) \le -1.$ (e.g., see Srivastava and Karlsson [10, p. 18]).

The symbol $(a)_n$ in the series (2.5) represents the Pochhammer symbol or shifted factorial and is defined by (see Srivastava and Karlsson [10, (2),(3) p. 16])

$$(\lambda)_n = \begin{cases} 1, & \text{if } n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if } n = 1,2,3,\cdots \end{cases} \qquad (2.6)$$

and also

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \lambda \ne 0,-1,-2,\cdots \qquad (2.7)$$

Besides these results, the following property of the shifted factorial (Srivastava and Karlsson [10, (9), p. 17]would also be required in proving the main result of this paper in section 3:

$$(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n \qquad (2.8)$$

Definition 2.3: The Humbert's confluent hypergeometric function Φ_1 is defined as (see, for example, Srivastava and Karlsson [10, (16), p. 25])

$$\Phi_1[\alpha,\beta;\gamma;x,y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!} \qquad (2.9)$$

where, $|x|<1, |y|<\infty$.

This Humbert's function along with the other Humbert's functions were generalized to the case of matrix arguments, first of all by Mathai [11, 12, 13] who gave a number of results for this function and later on, by this author [14, 15, 16].

3. Laplace Transform of the Generalized Incomplete Gamma Function

Now we state the main result of the paper, which gives the Laplace transform of the generalized incomplete gamma function (1.5) in terms of the Humbert function Φ_1 as the theorem below:

Theorem 3.1:

$$L\left\{b^{\nu-1}\gamma(\alpha,x;b);s\right\}$$
$$=\frac{x^{\alpha+\nu}\Gamma(\nu)}{\alpha+\nu}\Phi_1\left[\alpha+\nu,\nu;\alpha+\nu+1;-sx,-x\right] \qquad (3.1)$$

Proof: To prove this result, we apply the definition of the Laplace transform of a function (2.1) to the function $b^{\nu-1}\gamma(\alpha,x;b)$ to obtain

$$L\left\{b^{\nu-1}\gamma(\alpha,x;b);s\right\}$$
$$=\int_0^\infty e^{-sb}b^{\nu-1}\gamma(\alpha,x;b)\,db \qquad (3.2)$$

Consider the definition of the generalized incomplete gamma function (1.5). In this equation we put $t=xy$, so that, $dt=xdy$ which renders (1.5) as below

$$\gamma(\alpha,x;b)=x^\alpha\int_0^1 e^{-xy-bx^{-1}y^{-1}}y^{\alpha-1}dy \qquad (3.3)$$

Now replacing the function $\gamma(\alpha,x;b)$ in the integrand on the on the right side of (3.2) by its value, as given by (3.3), we have

$$L\left\{b^{\nu-1}\gamma(\alpha,x;b);s\right\}$$
$$=\int_0^\infty e^{-sb}b^{\nu-1}\left[x^\alpha\int_0^1 e^{-xy-bx^{-1}y^{-1}}y^{\alpha-1}dy\right]db \qquad (3.4)$$

Since the two integrals involved in the right side of (3.4) are uniformly convergent in their range of integration, their order can be changed to write (3.5) as

$$L\left\{b^{\nu-1}\gamma(\alpha,x;b);s\right\}$$
$$=x^\alpha\int_0^1 e^{-xy}y^{\alpha-1}\left[\int_0^\infty e^{-\left(s+x^{-1}y^{-1}\right)b}b^{\nu-1}db\right]dy \qquad (3.5)$$

The integral in b inside the square brackets on the right side of (3.5) can now be readily written in terms of a gamma function by the help of the following very well known result (e.g., see Erdélyi A., Magnus W., Oberhettinger F., Tricomi F. G. Vol I [4, 1.1 (5), p. 1]) which is a simple generalization of (1.1):

$$\Gamma(z)=s^z\int_0^\infty e^{-st}t^{z-1}dt, \qquad \mathrm{Re}(z)>0 \qquad (3.6)$$

In the light of (3.6), the right side expression in (3.5) yields

$$L\left\{b^{\nu-1}\gamma(\alpha,x;b);s\right\}$$
$$=x^\alpha\Gamma(\nu)\int_0^1 e^{-xy}y^{\alpha-1}\left(s+x^{-1}y^{-1}\right)^{-\nu}dy \qquad (3.7)$$

which may also be rewritten as

$$L\left\{b^{\nu-1}\gamma(\alpha,x;b);s\right\}$$

$$= x^{\alpha+\nu}\Gamma(\nu)\int_0^1 e^{-xy} y^{\alpha+\nu-1}(1+sxy)^{-\nu}\, dy \qquad (3.8)$$

The problem that remains now is to evaluate the integral in y on the right side of (3.8). We denote this integral by I, thus

$$I = \int_0^1 e^{-xy} y^{\alpha+\nu-1}(1+sxy)^{-\nu}\, dy \qquad (3.9)$$

needs to be evaluated. For this purpose we write the infinite series expansion of the exponential term e^{-xy} as

$$e^{-xy} = \sum_{n=0}^{\infty}\frac{(-xy)^n}{n!} \qquad (3.10)$$

and, since the series and concerned integral are uniformly convergent in their range of summation and integration respectively, we can interchange the order of summation and integration to rewrite (3.9), with the aid of (3.10) as below

$$I = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\left\{\int_0^1 y^{\alpha+\nu+n-1}(1+sxy)^{-\nu}\, dy\right\} \qquad (3.11)$$

The integral within the parentheses on the right of (3.11) can be written in terms of an incomplete beta function with the help of (2.3), as

$$I = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\frac{1}{(-sx)^{\alpha+\nu+n}} B_{-sx}(\alpha+\nu+n, 1-\nu) \qquad (3.12)$$

Utilizing the result of (2.4) for the incomplete beta function $B_{-sx}(\alpha+\nu+n,1-\nu)$ in (3.12) yields

$$I = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\frac{1}{\alpha+\nu+n}\times$$

$$\left\{{}_2F_1(\alpha+\nu+n,\nu;\alpha+\nu+n+1;-sx)\right\} \qquad (3.13)$$

which on expanding the ${}_2F_1$ function with the help of (2.5) yields

$$I = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\alpha+\nu+n}\frac{(\alpha+\nu+n)_m(\nu)_m}{(\alpha+\nu+n+1)_m}\frac{(-x)^n}{n!}\frac{(-sx)^m}{m!} \qquad (3.14)$$

Now, with the help of (2.8) and (2.7) it can be seen that

$$(\alpha+\nu+n)(\alpha+\nu+n+1)_m = (\alpha+\nu+n)_{m+1}$$

$$= \frac{\Gamma(\alpha+\nu+n+m+1)}{\Gamma(\alpha+\nu+n)} = \frac{(\alpha+\nu)_{n+m+1}}{(\alpha+\nu)_n}$$

$$= \frac{(\alpha+\nu)(\alpha+\nu+1)_{n+m}}{(\alpha+\nu)_n} \qquad (3.15)$$

and

$$(\alpha+\nu+n)_m = \frac{\Gamma(\alpha+\nu+n+m)}{\Gamma(\alpha+\nu+n)} = \frac{(\alpha+\nu)_{n+m}}{(\alpha+\nu)_n} \qquad (3.16)$$

Using (3.15) and (3.16) in (3.14) leads us to

$$I = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha+\nu)_{n+m}(\nu)_m}{(\alpha+\nu)(\alpha+\nu+1)_{n+m}}\frac{(-x)^n}{n!}\frac{(-sx)^m}{m!} \qquad (3.17)$$

which when interpreted in terms of (2.9) generates

$$I = \frac{1}{\alpha+\nu}\Phi_1[\alpha+\nu,\nu;\alpha+\nu+1;-sx,-x] \qquad (3.18)$$

Substitution of the above value of I in (3.8) thus proves (3.1).

As stated earlier, the result in (3.1) has already been generalized to the case of the corresponding functions of matrix arguments by this author in a further communication [1] which will appear in the near future.

The special case of (3.1) corresponding to $\nu = 1$, in fact, gives the Laplace transform of the generalized incomplete gamma function $\gamma(\alpha, x; b)$ in terms of the Humbert function Φ_1.

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