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n - NORMAL ALMOST DISTRIBUTIVE LATTICES

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Abstract:
The concept of *n*-normal lattices was introduced by William H. Cornish[7]. We defined an *n*-prime ideal in an ADL *R* and characterized an *n* - prime ideal in terms of distinct prime ideals of *R*. An *n*-normal ADL is defined and it is proved that an ADL *R* is an *n*-normal ADL if and only if for each prime ideal *P* of *R*, P^0 is an *n*+1 prime ideal of *R*. Also, we characterized an *n*-normal ADL in terms of its minimal prime ideals and also in terms of its annihilators. A sectionally *n*-normal ADL is defined in a natural way and it is proved that an ADL *R* is *n*-normal if and only if *R* is sectionally *n*-normal. Finally, it is proved that every ideal in an ADL *R* is *n*-normal as a sub ADL of *R* if and only if *R* is *n*-normal.

Keywords: Almost Distributive Lattice(ADL), n-prime ideal, n-Normal ADL.

Introduction

The concept of an Normal Almost Distributive lattice(ADL) was introduced in [5] as a common abstraction of all existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. In [7], W.H.Cornish introduced the concept of normal lattices and relatively normal lattices and studied their properties. In[5], we defined an ADL to be normal if its principal ideal lattice is a normal lattice.

The concept of *n* - normal lattices was introduced by William. H. Cornish [7]. An *n* - normal lattice is a distributive lattice with '0' in which every prime ideal contains at most *n* minimal prime ideals.

In this paper we introduce the concept of an *n*-normal ADL and study some of its properties. Also, we characterize an *n*-normal ADL in terms of its minimal prime ideals and also in terms of its annihilators.

We characterized the normal ADLs in terms of its prime ideals and minimal prime ideals. We defined the concept of sectionally normal and proved that every normal ADL is sectionally *n*-normal and conversely. Finally, we characterized *n*- normal ADLs in terms of its prime ideals.

Throughout this paper the letter *R* stands for an ADL $(R, \vee, \wedge, 0)$ with 0.

0. Preliminaries

An Almost Distributive Lattice (ADL)[3] is an algebra $(R, \vee, \wedge)$ of type (2, 2) satisfying

1: $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

2: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

3: $(x \vee y) \wedge y = y$

4: $(x \vee y) \wedge x = x$

5: $x \vee (x \wedge y) = x$

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in R$ such that $0 \wedge a = 0$ for all $a \in R$, then $(R, \vee, \wedge, 0)$ is called an ADL with 0. An ADL $(R, \vee, \wedge, 0)$ satisfies many properties satisfied by a distributive lattice with 0. These results are taken from [3],[5].

I). Let $(R, \vee, \wedge, 0)$ be an ADL with 0. Then for any $a \in R$,

1) : $a \vee 0 = a$

2) : $0 \vee a = a$

$3): a \wedge 0 = 0$

$4): a \wedge a = a$

$5): a \vee a = a$

II). Let $(R, \vee, \wedge, 0)$ be an an ADL with 0. Then it satisfies the following:

$1): (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

$2): a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$3): a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$4): (a \vee b) \wedge b = b$

$5): 0 \wedge a = 0$

$6): a \vee 0 = a$ for all $a, b, c \in R$

III). If $(R, \vee, \wedge, 0)$ is an ADL, for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$) , then $\leq$ is a partial ordering on R and for any $a, b, c \in R$, we have the following:

$(1): a \vee b = a \Leftrightarrow a \wedge b = b$

$(2): a \vee b = b \Leftrightarrow a \wedge b = a$

$(3): a \vee b = b \vee a$ whenever $a \leq b$

$(4): \wedge$ is associative in R.

$(5): a \wedge b \wedge c = b \wedge a \wedge c$

$(6): (a \vee b) \wedge c = (b \vee a) \wedge c$

$(7): a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

$(8): a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$(9): a \wedge (a \vee b) = a; (a \wedge b) \vee b = b;$ and $a \vee (b \wedge a) = a$

$(10): a \leq a \vee b$ and $a \wedge b \leq b$

$(11): a \wedge a = a$ and $a \vee a = a$

$(12): 0 \vee a = a$ and $a \wedge 0 = 0$

$(13):$ *If* $a \leq c;\ b \leq c$ then $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

$(14): a \vee b = a \vee b \vee a$

It can be observed that an ADL *R* satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$ and an absorption law. Any one of these properties make an ADL *R* a distributive lattice. That is

0.1. Theorem [5] : Let $(R, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent.

$1): (R, \vee, \wedge, 0)$ is a distributive lattice.

$2): a \vee b = b \vee a,$ for all $a, b \in R.$

$3): a \wedge b = b \wedge a,$ for all $a, b \in R..$

$4): (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, for all $a, b, c \in R.$

As usual, an element $m \in R$ is called maximal if it is maximal element in the partially ordered set $(R, \leq).$ That is, for any $a \in R, m \leq a \Rightarrow m = a.$

0.2. Theorem [5]: Let R be an ADL and $m \in R$. Then the following are equivalent.

$1): m$ is maximal with respect to $\leq$.

$2): m \vee a = m,$ for all $a \in R$

$3): m \wedge a = a,$ for all $a \in R$

$4): a \vee m$ is maximal, for all $a \in R$

As in distributive lattices ([1],[2]), we define a non-empty sub set I of R is an ideal of R if for any $a,\ b \in I, a \vee b \in I$ and $a \wedge x \in I$ for any $a \in I, x \in R.$
Also, a non-empty subset F of R is said to be a filter of R if for every $a,\ b \in F, a \wedge b \in F$ and $x \vee a \in F$ for $a \in F, x \in R.$
The set $I(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion such that, for any $I,\ J \in I(R), I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J = \{a \vee b \mid a \in I,\ b \in J\}.$
A proper ideal P of R is a prime ideal if for any $x,\ y \in R,\ x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of R is said to be maximal if it is not properly contained in any proper ideal of R. It can be observed that every maximal ideal F of R is a prime ideal. Every proper ideal of R is contained in a maximal ideal. For any sub set S of R the smallest ideal containing *S* is given by
$(S] = \{(\vee^{n}_{i=1} s_i) \wedge x \mid s_i \in S,\ x \in R$ and $n \in N\}.$

0.3. Lemma [5]: For any x, y in R the following are equivalent

1). $(x] \subseteq (y]$
2). $y \wedge x = x$
3). $y \vee x = y$
4). $[y) \subseteq [x)$

For any $x,\ y \in R,$ it can be verified that $(x] \vee (y] = (x \vee y]$ and $(a] \wedge (b] = (a \wedge b].$ Hence the set $PI(R)$ of all principal ideals of R is a sub lattice of the distributive lattice $I(R)$ of ideals of R. Hence several important concepts/properties of distributive lattices can be introduced in the class of ADLs through its principal ideal lattices.
A prime ideal P of R is said to be minimal if it is minimal among all the prime ideals of R

0.5. Theorem [4]: Every prime ideal of R contains a minimal prime ideal. Let P be a prime ideal containing an ideal I of R. Then P is a minimal prime ideal belonging to I iff for each $x \in P$ there is a $y \notin P$ such that $x \wedge y \in I.$
A prime ideal *P* of *R* is a minimal prime ideal if and only if for each $x \in P,$ there is $y \notin P$ such that $x \wedge y = 0.$
A lattice with 0 is called normal if each prime ideal contains a unique minimal prime ideal.

0.6. Definition: Let P be an ideal in R. Then we define $P^0 = \{x \in R \mid x \wedge a = 0$, for some $a \in R - P\}$.

0.7. Theorem [5]: *If P is a prime ideal in R, then the ideal P^0 is the intersection of all the minimal prime ideals contained in P.*

1. n - Prime Ideals in an ADL

We begin this section with the following definition.
1.1. Definition : Let *A* be any nonempty subset of an ADL R and $a \in R.$ Then we define
$A_a = \{x \in R \mid x \wedge a \in A\}.$ We call A_a as a section of A by $a.$

First we prove the following lemma.
1.2. Lemma : Let A, B be any two subsets of an ADL R and $a \in R.$. Then

1). $A \subseteq B \Rightarrow A_a \subseteq B_a$
2). $(A \cap B)_a = A_a \cap B_a$
3). $(A \cup B)_a = A_a \cup B_a$

In the following result we give a relation between an ideal of an ADL R and a section of an ideal I by an element of R.

1.3. Theorem : Let I be any ideal in an ADL R and $a \in R.$ Then I_a is an ideal of R and $I \subseteq I_a$

Proof : Let I be an ideal of an ADL R and $a \in R.$. Since $0 \in I,$ we have $0 \wedge a = 0 \in I$ and hence $0 \in I_a$. Therefore $I_a \neq \phi.$ Let $x, y \in I_a$. Then $x \wedge a \in I$ and $y \wedge a \in I$. Since I is an ideal of R, we have $(x \wedge a) \vee (y \wedge a) \in I$ and hence $(x \vee y) \wedge a \in I.$ Therefore $x \vee y \in I_a.$ Again, let $x \in I_a$. Then $x \wedge a \in I$. Since I is an ideal, for any $r \in R, x \wedge a \wedge r \in I.$ This gives $r \wedge x \wedge a \in I$ and hence $x \wedge r \wedge a \in I.$ Therefore $x \wedge r \in I_a$. Thus I_a is an ideal of R.
Let $x \in I$. Since I is an ideal of R, for $a \in R, x \wedge a \in I$. Therefore $x \in I_a$ and hence $I \subseteq I_a$.

Now we prove that if a is any element of an ideal I of R, then the section of I by a is R itself.

1.4. Theorem : Let I be an ideal of an ADL R and $a \in R$. Then $a \in I \Leftrightarrow I_a = R$

Proof : Let I be an ideal of an ADL R and $a \in I.$ Since I is an ideal of R, for any $x \in R, a \wedge x \in I$ and hence $x \wedge a \in I.$ Then from the definition of $I_a,$ we get $x \in I_a.$ Therefore we get $R \subseteq I_a.$ Thus we have $R = I_a.$

Conversely, suppose that $I_a = R$. Since $a \in R$, we get $a \in I_a$ and hence $a \wedge a \in I$. Therefore $a \in I.$

1.5. Corollary : Let I be an ideal of an ADL R. Then $I_0 = R.$

If R is an ADL with a maximal element and I is an ideal in R, then in the following result we we can observe that the section of I by a maximal element is the ideal I itself.

1.6. Lemma : If $m \in R$ is a maximal element of an ADL R, then $I_m = I.$

In the following result we prove that if P is a prime ideal of an ADL R, then the section of P by an element which is not in P is P itself.

1.7. Lemma : If P is a prime ideal of an ADL R then for any $a \notin P, P_a = P.$

Proof : Let P be a prime ideal of an ADL R and $a \notin P.$ Since P is an ideal of R, from Theorem 1.3, we have $P \subseteq P_a$. Let $x \in P_a$. Then $x \wedge a \in P$. Since P is a prime ideal of R, either $x \in P$ or $a \in P$. But $a \notin P.$ Therefore $x \in P.$ Hence $P = P_a$.

Now we prove the following theorem.

1.8. Theorem : Let I, J be any two ideals of an ADL R and $a, b \in R.$ Then

1). $(I_a)_a = I_a$

2). $(I_a)_b = I_{b \wedge a} = I_{a \wedge b} = (I_b)_a$

3). $I_a \cap I_b = I_{a \vee b}$

4). $I_a \vee I_b \subseteq I_{a \wedge b}$

5). $(I \vee J)_a = (I_a \vee J_a)_a$

6). $(I \wedge J)_a = (I_a \wedge J_a)_a$

1.11. Definition : Let R be an ADL and A, B be any two subsets of R. Then A is said to be pair wise in B if for every $x, y \in A$ with $x \neq y, x \wedge y \in B.$ The concept of an $n-$ prime ideal in a distributive lattice was introduced by William. H. Cornish [7]. Analogously we define an $n-$ prime ideal in an ADL as follows.

1.12. Definition : An ideal I of an ADL R is said to be $n-$ prime ideal of R if for any $x_1, x_2, \ldots\ldots, x_n$ in R with $x_i \wedge x_j \in I$ for all $i \neq j$ then $x_k \in I$ for some $k \in \{1, 2, \ldots\ldots, n\}.$

By routine verification using the definition of an $n-$prime ideal[1.12], we can obtain the following lemma.

1.13. Lemma : Let I be an ideal of an ADL R. Then

1). Every ideal of R is a 1- prime ideal.

2). I is a prime ideal if and only if I is a 2 - prime ideal.

3). For $2 \le n \le m,$ every $n-$ prime ideal of R is also an $m-$ prime ideal of R.

1.14. Lemma : Let I be an ideal of an ADL R. If m is the largest integer such that there exist $a_1,\ a_2,\ .....,\ a_m$ in $R-I$, which are pair wise in I then for $1 \le i \le m$, I_{a_i} is a prime ideal of R.

Proof : Let I be an ideal of an ADL R. Let m be the largest integer such that $a_1,\ a_2,\ \ldots..a_m$ be m elements of $R-I$, which are pair wise in I. From Theorem 1.3, we have I_{a_i}, where $1 \le i \le m,$ is an ideal of R. Now we prove that I_{a_i} is a prime ideal of R.

If $a_i \in I_{a_i}$ then $a_i \wedge a_i \in I.$ That is $a_i \in I$. This is not possible. Thus $a_i \notin I_{a_i}$. Therefore I_{a_i} is a proper ideal of R. Let $x,\ y \in R$ and $x \wedge y \in I_{a_i}$. Then $x \wedge y \wedge a_i \in I$ and hence $(x \wedge a_i) \wedge (y \wedge a_i) \in I.$
Now consider the set of (m+1) elements $\{a_1,\ a_2,\ ....,\ a_{i-1}, (x \wedge a_i), (y \wedge a_i),\ a_{i+1},\ ...., a_m\}$ of R.
Clearly this set is pairwise in I and containing (m + 1) elements. Therefore from the hypothesis, we get either $x \wedge a_i \in I$ or $y \wedge a_i \in I$. This gives either $x \in I_{a_i}$ or $y \in I_{a_i}$. Therefore I_{a_i} is a prime ideal of R.

In the following theorem we characterize an n - prime ideal of R in terms of ideals.

1.15. Theorem : Let I be an ideal in ADL R and $n \ge 2$. Then I is $n-$ prime ideal of R if and only if for any ideals $I_1,\ I_2,\ ...,\ I_n$ in R such that $I_i \wedge I_j \subseteq I, i \ne j,$ there exists k such that $I_k \subseteq I.$

Proof : Let I be an ideal of an ADL R. Assume that I is an $n-$prime ideal of R.
Let $I_1, I_2,\ .....,\ I_n$ be the set of n ideals in R which are pair wise in I.
Suppose I_k is not contained in I, for all $k = 1, 2,\ ....,\ n.$.
Now choose $x_k \in I_k$ such that $x_k \notin I$ for $1 \le k \le n.$ Since $I_i \wedge I_j \subseteq I,$ we get $x_i \wedge x_j \in I$ for all $i \ne j.$ Since I is $n-$ prime ideal of R, we get $x_k \in I$ for some $k \in \{1, 2, ..., n\}.$ This is a contradiction. Therefore there exists some $k \in \{1, 2, ..., n\}$ such that $I_k \subseteq I.$
Conversely, assume the given condition.
We have to prove that I is an $n-$prime ideal of R. Let $\{x_1, x_2,\ ......, x_n\}$ be any subset of R which is pair wise in I. Now we prove that there exist some $k \in \{1, 2, ..., n\}$ such that $x_k \in I.$
Now consider the ideals $\{(x_1], (x_2],\ ....., (x_n]\}$ of R.
Since $x_i \wedge x_j \in I,$ for $i \ne j$, we get $(x_i] \wedge (x_j] \subseteq I$ for $i \ne j$. Then from our assumption, there exists $k \in \{1, 2, ..., n\}$ such that $(x_k] \subseteq I$ and hence we get $x_k \in I.$ Therefore I is an $n-$prime ideal of R

Now we conclude this section with the following theorem in which we characterize an $n-$prime ideal of R in terms of prime ideals of R.

1.16. Theorem : Let I be an ideal in an ADL R. Then I is an $n-$ prime ideal of R if and only if I is the intersection of at most n − 1 distinct prime ideals of R.

Proof : Let I be an ideal of an ADL R. Assume that I is an $n-$prime ideal of R.
If n = 2, then from (2) of Lemma 1.13, I becomes a prime ideal and hence the required condition is trivial.
Now assume that $m < n$ is the largest integer such that for any $x_1, x_2, ..., x_n$ of R are pair wise in I and $x_i \notin I$ for all $i \in \{1, 2, ..., n\}.$ Then from Lemma 1.14, I_{x_i} becomes a prime ideal of R. Also, from Lemma 1.3, $I \subseteq I_{x_i}$ for all $i \in \{1, 2, ..., n\}.$ Therefore $I \subseteq \bigcap_{i=1}^{m} I_{x_i}.$

Again let $a \in \bigcap_{i=1}^{m} I_{x_i}$. Then $a \in I_{x_i}$ for all $i \in \{1,2,....., m\}$. Then $a \wedge x_i \in I$ for all $i \in \{1,2,....., m\}$. Therefore we get the set $\{a, x_1, x_2, ..., x_m\}$ is pair wise in I .Since this set containing m+1 elements of R and $x_i \notin I$ for all i, from assumption we get, $a \in I$. Therefore $\bigcap_{i=1}^{m} I_{x_i} \subseteq I$. Thus $I = \bigcap_{i=1}^{m} I_{x_i}$.

That is, I is the intersection of m ($m < n$) prime ideals of R.

Conversely, assume that $P_1, P_2, ..., P_k$ $(1 \le k \le n-1)$ are distinct prime ideals of R such that $I = P_1 \cap P_2 \cap ..... \cap P_k$. We have to prove that I is a $n-$prime ideal of R.

Let $\{x_1, x_2, ..., x_n\}$ be any subset of R which is pair wise in I.

Suppose $x_i \notin I$ for $1 \le i \le n$. Then for each r for $1 \le r \le k$, there is at most one i $(1 \le i \le k)$ such that $x_i \notin P_r$. Since $k < n$, there is some r such that $x_i \notin P_r$. for two distinct i's. This is a contradiction. Therefore, there exists some $k \in \{1,2,....., n-1\}$ such that $x_k \in I$.

Hence I is an $n-$prime ideal of R.

2. n - Normal ADLs

The concepts of $n-$normal lattices and sectionally $n-$normal lattices were introduced by William.H.Cornish in [7]. In this section we define an $n-$normal ADL and study some of its properties.

Now we begin this section with the following definition.

2.1. Definition : Let $n \ge 1$ be a positive integer. An ADL R with 0 is called $n-$normal if each prime ideal in R contains at most n minimal prime ideals of R.

2.2. Theorem : Let R be an ADL with 0 and n be a positive integer. Then R is $n-$normal if and only if for each prime ideal P , P^0 is an $(n+1)-$ prime ideal.

Proof : Let P be a prime ideal of a $n-$normal ADL R. Then P contains at most n minimal prime ideals of R. From Theorem 0.7, P^0 is the intersection of all minimal prime ideals of R contained in P. Since P contains at most n minimal prime ideals of R, P^0 is the intersection of at most n minimal ideals. Therefore from 1.16, P^0 is an $(n+1)-$prime ideal of R. Conversely, suppose, for each prime ideal P of R, P^0 is a $(n+1)-$prime ideal. We have to prove that R is $n-$normal.

Now we prove that each prime ideal of R contains at most n minimal prime ideals of R. Let P be a prime ideal of R. Then P^0 is an $(n+1)-$prime ideal. Then from Theorem 1.16, P^0 is the intersection of at most n distinct prime ideals of R. Let $P_1, P_2, ..., P_k, 1 \le k \le n,$ be distinct prime ideals of R such that $P_0 = P_1 \cap P_2 \cap :::: \cap P_k = \bigcap_{i=1}^{k} P_i$. Let Q be a minimal prime ideal contained in P. Since P^0 is the intersection of all the minimal prime ideals contained in P, we get $P^0 \subseteq Q \subseteq P$. That is $\bigcap_{i=1}^{k} P_i \subseteq Q \subseteq P$. Since Q is a minimal prime ideal, we get $P_i = Q$ for some $i (1 \le i \le k)$. Thus the minimal prime ideals contained in P are among $\{P_1, P_2, ..., P_k\}$. Hence P contains at most $k \le n$ minimal prime ideals. That is every prime ideal of R contains at most n minimal prime ideals of R. Therefore R is an $n-$normal ADL.

Now we characterize an $n-$normal ADL in terms of its minimal prime ideals.

2.3. Theorem : An ADL R is $n-$normal if and only if for any $(n+1)$ distinct minimal prime ideals $Q_0, Q_1, \ldots\ldots, Q_n$ of R, $\vee_{i=0}^{n} Q_i = R.$

Proof : Let R be an $n-$normal ADL. Then every prime ideal of R contains atmost n minimal prime ideals of R. Suppose $\vee_{i=0}^{n} Q_i \neq R.$. Then $\vee_{i=0}^{n} Q_i$ is a proper ideal of R, hence it is contained in a prime(maximal) ideal P of R. Since R is $n-$normal each prime ideal of R contains at most n minimal prime ideals. Therefore, our supposition is wrong. Hence $\vee_{i=0}^{n} Q_i = R.$.

Conversely, assume the given condition. Let P be any prime ideal of R. We have to prove that P contains at most n minimal prime ideals of R. Suppose P contains $(n+1)$ distinct minimal prime ideals say $Q_0, Q_1, \ldots\ldots, Q_n$ of R. Then $\vee_{i=0}^{n} Q_i \subseteq P.$ But from our assumption, we have $\vee_{i=0}^{n} Q_i = R.$. This gives $R \subseteq P$. Since P is a proper ideal of R, this is a contradiction. Therefore every prime ideal of R contains at most n minimal prime ideals of R. Hence R is $n-$normal.

In the following theorem we characterize an $n-$normal ADL in terms of its annihilators.

2.4. Theorem : An ADL R is $n-$normal if and only if for any $x_0, x_1, \ldots\ldots, x_n \in R,$ such that $x_i \wedge x_j = 0$ for $i \neq j$ where $0 \leq i \leq n$ and $0 \leq j \leq n$, $\vee_{i=0}^{n} (x_i)^* = R$.

Proof : Assume that R is an $n-$normal ADL. Let $x_0, x_1, \ldots\ldots, x_n$ be (n+1) elements of R such that $x_i \wedge x_j = 0$, for $i \neq j$. Suppose $\vee_{i=0}^{n} (x_i)^* \neq R.$ Then $\vee_{i=0}^{n} (x_i)^*$ is a proper ideal of R and hence $\vee_{i=0}^{n} (x_i)^*$ is contained in a maximal ideal say M of R. That is $(x_0)^* \vee (x_1)^* \vee \ldots \vee (x_n)^* \subseteq M$. Since M is a prime ideal and $(x_i)^* \subseteq M$ by the 0.6 definition [7]we get $x_i \notin M^0$. Now, since R is an $n-$normal ADL and M is a prime ideal of R ,from Theorem 2.2, we have M^0 is (n + 1) prime ideal of $R.$

Conversely, assume the given condition. We have to prove that R is an $n-$normal ADL. Let P be any prime ideal of R. Then from Theorem 2.2, it is enough to prove that P^0 is an $(n+1)-$prime ideal of R. Let $a_0, a_1, \ldots\ldots, a_n$ be (n+1) elements of R which are pair wise in P^0 . Since $a_i \wedge a_j \in P^0$ for $i \neq j,$ from the definition of P^0, there exists $y_{ij} \in R-P$ such that $(a_i \wedge a_j) \wedge y_{ij} = 0.$

Now, for $0 \leq i \leq n,$ consider the elements $x_i = a_i \wedge (y_{i1} \wedge y_{i2} \wedge \ldots\ldots \wedge y_{i(i-1)} \wedge y_{i(i+1)} \wedge \ldots\ldots \wedge y_{in}).$

Clearly, for $i \neq j$ we have $x_i \wedge x_j = 0.$ Therefore from our assumption, we get $\vee_{i=0}^{n} (x_i)^* = R$.

If $(x_k)^* \subseteq P$ for all k , then we get $R = \vee_{i=0}^{n} (x_i)^* \subseteq P$: This is a contradiction. Therefore $(x_k)^*$ is not contained in P for some $k (0 \leq k \leq n)$ and hence $x_k \in P^0.$ Then by the definition of P^0, there exists $a \in R-P$ such that $x_k \wedge a = 0.$

$$\text{Now} \quad x_k \wedge a = 0 \Rightarrow (a_k \wedge y_{k1} \wedge y_{k2} \wedge \ldots\ldots \wedge y_{k(k-1)} \wedge y_{k(k+1)} \wedge \ldots\ldots \wedge y_{kn}) \wedge a = 0$$
$$\Rightarrow a_k \wedge (y_{k1} \wedge y_{k2} \wedge \ldots\ldots \wedge y_{k(k-1)} \wedge y_{k(k+1)} \wedge \ldots\ldots \wedge y_{kn} \wedge a) = 0$$

Since $a \in R-P$ and $y_{k1}, y_{k2} \ldots\ldots, y_{kn} \notin P,$ we have $(y_{k1} \wedge y_{k2} \wedge \ldots\ldots \wedge y_{kn} \wedge a) \in R-P.$

Therefore $a_k \in P^0$ (by definition of P^0) and hence P^0 is a $(n+1)-$prime ideal of R.

2.5. Corollary : Let R be a $n-$normal ADL. Then for any (n + 1) elements $x_0, x_1, \ldots\ldots, x_n$ of R ,

$$(\wedge_{i=0}^{n} x_i)^* = \vee_{i=0}^{n} (x_0 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots\ldots \wedge x_n)^*$$

Proof : Let R be an $n-$normal ADL and $x_0, x_1, \ldots\ldots, x_n$ be (n + 1) elements of R.

Let $a = x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_n$ and $b_i = x_0 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots\ldots \wedge x_n$ for $0 \leq i \leq n.$

We have to prove that $(a)^* = \vee_{i=0}^{n} (b_i)^*$

Let $x \in (a)^*$. Then $x \wedge a = 0.$

Now for $i \neq j$, we have $b_i \wedge b_j = x_0 \wedge x_1 \wedge \ldots\ldots \wedge x_n = a$ and hence $(x \wedge b_i \wedge b_j) = x \wedge a = 0$.
That is $(x \wedge b_i) \wedge (x \wedge b_j) = 0$ for $i \neq j$.
Now consider the set $\{x \wedge b_0, x \wedge b_1 \ldots\ldots, x \wedge b_n\}$.
Since R is $n-$normal and $(x \wedge b_i) \wedge (x \wedge b_j) = 0$, for $i \neq j$, from Theorem 2.4, we have $R = \bigvee_{i=0}^{n} (x \wedge b_i)^*$. Therefore $x \in R$ can be written as $x = \bigvee_{i=0}^{n} a_i$ where $a_i \in (x \wedge b_i)^*$ for all i.
Then $a_i \wedge x \wedge b_i = 0$ and hence we get $a_i \wedge x \in (b_i)^*$ for all i.
Now $x = x \wedge x = (\bigvee_{i=0}^{n} a_i) \wedge x = \bigvee_{i=0}^{n} (a_i \wedge x) \in \bigvee_{i=0}^{n} (b_i)^*$
Therefore $x \in \bigvee_{i=0}^{n} (b_i)^*$. Thus we get $(a)^* \subseteq \bigvee_{i=0}^{n} (b_i)^*$.
Now let $x \in (b_i)^*$. for some i. Then $x \wedge b_i = 0$ and hence $x_i \wedge x \wedge b_i = 0$.
That is $x \wedge x_i \wedge x_1 \wedge \ldots\ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots\ldots x_n = 0$ and hence we get $x \wedge a = 0$.
Therefore $x \in (a)^*$. Thus we get $(b_i)^* \subseteq (a)^*$. Therefore $\bigvee_{i=0}^{n} (b_i)^* \subseteq (a)^*$.
Hence $(a)^* = \bigvee_{i=0}^{n} (b_i)^*$

Analogous to the concept of sectionally normal ADL, now we introduce the concept of sectionally $n-$normal ADL.

2.6. Definition : Let $n \geq 1$ be a positive integer. An ADL R with '0' is called sectionally $n-$normal if each $x \in R$, the interval $[0\ x]$ is an $n-$normal lattice.
Now we recall that, for any nonempty subset S of R, $(x)_S^* = (x)^* \cap S = \{t \in S \mid t \wedge x = 0\}$
We conclude this section with the following theorem in which, we prove that an ADL R is $n-$normal if and only if R is sectionally $n-$normal. Since every ideal of R is a sub ADL, we also prove that an ADL R is $n-$normal if and only if every ideal of R is $n-$normal as a sub ADL of R.

2.7. Theorem : The following conditions are equivalent for an ADL R with '0'.

1). R is $n-$normal.
2). Each ideal I of R is an $n-$normal as a sub ADL of R.
3). R is sectionally $n-$normal.

Proof :
(1) ⇒ (2) : Assume that R is an $n-$normal ADL and I is an ideal of R.
We have to prove that I is $n-$normal as sub ADL of R. Let $x_0, x_1 \ldots\ldots x_n \in I$ such that $x_i \wedge x_j = 0$, for $i \neq j$. Since R is $n-$normal and $x_0, x_1 \ldots\ldots x_n \in R$ with $x_i \wedge x_j = 0$ for $i \neq j$, by Theorem 2.4 we have $R = \bigvee_{i=0}^{n} (x_i)^*$
Now $I = I \cap R = I \cap [\bigvee_{i=0}^{n} (x_i)^*] = \bigvee_{i=0}^{n} [I \cap (x_i)^*] = \bigvee_{i=0}^{n} (x_i)^*$
Therefore I is $n-$normal as a sub ADL of R.
(2) ⇒ (3) : Assume that each ideal of R is $n-$normal as a sub ADL of R.
Let $x \in R$ and $x \neq 0$. Now consider the interval $[0, x]$.
We have to prove that $[0\ x]$ is an $n-$normal lattice.
Let $x_0, x_1 \ldots\ldots x_n \in [0, x]$ such that $a_i \wedge a_j = 0$, for $i \neq j$. Then from Theorem 2.4, it is enough to prove that $[0, x] = \bigvee_{i=0}^{n} (a_i)_{[0,x]}^*$, where $(a_i)^*$ is an annihilator in $[0, x]$.
Let $t \in [0, x]$. Then $t \leq x$ and hence we get $t \wedge x = t$. This gives $t \in (x]$ and hence we get $[0, x] \subseteq (x]$. Clearly, $a_0, a_1, \ldots\ldots, a_n \in (x]$ and $a_i \wedge a_j = 0$, for $i \neq j$. Therefore from our assumption and Theorem 2.4, we get $(x] = \bigvee_{i=0}^{n} (a_i)^*$.

Now $[0,\ x] = [0,\ x] \cap (x]$

$= [0,\ x] \cap \{\bigvee_{i=0}^{n} (a_i)^*\}$

$= \bigvee_{i=0}^{n} \{[0, x] \cap (a_i)^*\}$ (from (4) of Theorem 1.15)

$= \bigvee_{i=0}^{n} (a_i)^*_{[0,x]}$,

Therefore $[0,\ x]$ is $n-$normal and hence R is sectionally $n-$normal.

(3) ⇒(1) : Assume that R is sectionally $n-$normal. We have to prove that R is $n-$normal.

Let $x_0, x_1, \ldots, x_n$ such that $x_i \wedge x_j = 0$ for $i \neq j$. Then from Theorem 2.4, it is enough to prove that $\bigvee_{i=0}^{n} (x_i)^* = R$

Clearly $\bigvee_{i=0}^{n} (x_i)^* \subseteq R.$

Now, let $a \in R.$ Then from our assumption, $[0,\ a]$ is an $n-$normal lattice.

Clearly, $(x_0 \wedge a), (x_1 \wedge a), \ldots, (x_n \wedge a) \in [0,\ a]$ and $(x_i \wedge a) \wedge (x_j \wedge a) = 0$ for $i \neq j.$

Therefore from Theorem 2.4, we get $[0,\ a] = \bigvee_{i=0}^{n} (x_i \wedge a)^*_{[0,a]}$,

Since $a \in [0,\ a],$ we can write $a = \bigvee_{i=0}^{n} t_i$ where $t_i \in (x_i \wedge a)^*.$

Now $a = a \wedge (\bigvee_{i=0}^{n} t_i) = \bigvee_{i=0}^{n} (a \wedge t_i).$

Since $t_i \in (x_i \wedge a)$_, we get $x_i \wedge a \wedge t_i = 0$ for all $i.$

This gives $a \wedge t_i \in (x_i)$_ for all i. Therefore $a \in \bigvee_{i=0}^{n} (x_i)^*$ and hence we get $R \subseteq \bigvee_{i=0}^{n} (x_i)^*$

Therefore $R = \bigvee_{i=0}^{n} (x_i)^*$

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