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Relatively Dually Normal ADL

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Abstract (10pt)

From the definition of an ADL it can be observed that an ADL R satisfies $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ …..(*) but does not satisfy the dual of (*), namely $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$. Thus the lattice theoretic duality principle does not hold in ADLs.

In this paper, we introduce the concept of relative dual annihilators in an ADL and study some of their properties. Also, we define a relatively Dually Normal ADL and we characterize a Relatively Dually Normal ADL R in terms of relative dual annihilators.

Keywords:

AlmostDistributive Lattice(ADL);
Relative dual annihilators;
RelativelyDually Normal ADL.

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Introduction :

The concept of an Almost Distributive lattice(ADL) was introduced in [3] as a common abstraction of all existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. In[2], we defined an ADL to be normal if its principal ideal lattice is a normal lattice and studied some of its properties. In [Dually Normal ADL], we introduced the concept of Dually Normal ADL and we characterized Dually Normal ADLs in terms of dual annihilators.

0. Preliminaries

An Almost Distributive Lattice (ADL) is an algebra $(R, \vee, \wedge)$ of type (2, 2) satisfying

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in R$ such that $0 \wedge a = 0$ for all $a \in R$, then $(R, \vee, \wedge, 0)$ is called an ADL with 0. As usual, an element $m \in R$ is called maximal if it is maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$.

Let R be an ADL and $m \in R$. Then the following are equivalent.

1). m is maximal with respect to $\leq$.

2). $m \vee a = m$, for all $a \in R$.

3). $m \wedge a = a$, for all $a \in R$.

4). $a \vee m$ is maximal, for all $a \in R$.

0.1. Definition : [4] Let A be a subset of an ADL R. and B is an ideal in R. Then we call the ideal $\lfloor A, B \rfloor = \{x \in R \mid x \wedge A \subseteq B\}$ as a relative annihilator of A with respect to B.

0.2. Theorem : [1] An ADL R. is normal if and only if every prime ideal of R. contains a unique minimal prime ideal of R.

0.3. Theorem : [2] Let R. be an ADL. Then R. is normal if and only if each prime filter in R. is contained in a unique maximal filter.

A relatively complemented ADL is an ADL in which every closed interval is a complemented lattice.

0.4. Theorem : [1] Every relatively complemented ADL is a normal ADL .

0.5. Theorem : [3] An ADL R. is dually normal if and only if for $x, y \in R$, if $x \vee y$ is maximal then there exist $x_1, y_1 \in R$ such that $x \vee x_1$, $y \vee y_1$ are maximal in R and $x_1 \wedge y_1 = 0$.

0.6. Theorem : [3] Let R be an ADL with maximal elements. Then the following are equivalent.

1). R is dually normal.

2). For any $x, y \in R$, if $x \vee y$ is maximal then $(x)^+ \vee (y)^+ = R$

3). For any $x, y \in R$, $(x \vee y)^+ = (x)^+ \vee (y)^+$

Note that, throughout this paper the letter R. stands for an ADL $(R, \vee, \wedge, 0)$.

1. Relative Dual Annihilators

In this section we introduce the concept of a relative dual annihilator, dual to that of relative annihilator defined in [4]. We study many important properties of a relative dual annihilator. Since the dual of an ADL is not an ADL, we provide proofs to all the results of relative dual annihilators in this section.
First we give the following definition which is dual to that of the definition 0.1.

1.1. Definition :

1). For any non-empty subset A of an ADL R and $x \in R, A \vee x = \{a \vee x \mid a \in R\}$

2). For any two subsets A, B of R define $\lceil A, B \rceil = \{x \in R \mid A \vee x \subseteq B\}$

By routine verification, we get the following result.

1.2. Lemma : For any subsets A, B of an ADL R. and $x \in R$, we have

1). $(A \cup B) \vee x = (A \vee x) \cup (B \vee x)$

2). $(A \cap B) \vee x \subseteq (A \vee x) \cap (B \vee x)$

Using Lemma 1.2, we prove the following lemma.

1.3. Lemma : Let A, B, C, D be any non-empty subsets of an ADL R. Then

1). $C \subseteq D \Rightarrow \lceil A, C \rceil \subseteq \lceil A, D \rceil$ and $\lceil D, A \rceil \subseteq \lceil C, A \rceil$.

2). $\lceil A, C \cup D \rceil = \lceil A, C \rceil \cup \lceil A, D \rceil$.

3). $\lceil A, C \cap D \rceil = \lceil A, C \rceil \cap \lceil A, D \rceil$.

4). $\lceil A \cup B, C \rceil = \lceil A, C \rceil \cap \lceil B, C \rceil$

5). $\lceil A,C \rceil \cup \lceil B,C \rceil \subseteq \lceil A \cap B,C \rceil$

Proof : 1). Let A, C, D be any three subsets of an ADL $R.$ and $C \subseteq D.$

Now, $x \in \lceil A,C \rceil \Rightarrow A \vee x \subseteq C$

$\Rightarrow A \vee x \subseteq D$ (since $C \subseteq D$)

$\Rightarrow x \in \lceil A,D \rceil$

Therefore $\lceil A,C \rceil \subseteq \lceil A,D \rceil$

Also, $x \in \lceil D,A \rceil \Rightarrow D \vee x \subseteq A$

$\Rightarrow C \vee x \subseteq D \vee x \subseteq A$ (since $C \subseteq D$)

$\Rightarrow C \vee x \subseteq A$

$\Rightarrow x \in \lceil C,A \rceil$. Therefore $\lceil D,A \rceil \subseteq \lceil C,A \rceil$

2). Let A, C, D be any three subsets of $R.$

Now, $x \in \lceil A,C \rceil \cup \lceil A,D \rceil \Leftrightarrow x \in \lceil A,C \rceil$ or $x \in \lceil A,D \rceil$

$\Leftrightarrow A \vee x \subseteq C$ or $A \vee x \subseteq D$

$\Leftrightarrow A \vee x \subseteq C \cup D$

$\Leftrightarrow x \in \lceil A,C \cup D \rceil$.

Therefore $\lceil A,C \rceil \cup \lceil A,D \rceil = \lceil A,C \cup D \rceil$.

3). Let A, C, D be any three subsets of $R.$

Now, $x \in \lceil A,C \rceil \cap \lceil A,D \rceil \Leftrightarrow x \in \lceil A,C \rceil$ and $x \in \lceil A,D \rceil$

$\Leftrightarrow A \vee x \subseteq C$ and $A \vee x \subseteq D$

$\Leftrightarrow A \vee x \subseteq C \cap D$

$\Leftrightarrow x \in \lceil A,C \cap D \rceil$.

Therefore $\lceil A,C \rceil \cap \lceil A,D \rceil = \lceil A,C \cap D \rceil$.

4). Let A, B, C be any three subsets of $R.$

Now, $x \in \lceil A \cup B,C \rceil \Leftrightarrow (A \cup B) \vee x \subseteq C$

$\Leftrightarrow (A \cup B) \vee x \subseteq C$

$\Leftrightarrow A \vee x \subseteq C$ and $B \vee x \subseteq C$

$\Leftrightarrow x \in \lceil A,C \rceil$ and $x \in \lceil B,C \rceil$

$\Leftrightarrow x \in \lceil A,C \rceil \cap \lceil B,C \rceil$.

Therefore $\lceil A \cup B,C \rceil = \lceil A,C \rceil \cap \lceil B,C \rceil$

5). Proof follows from condition (1).

In the following result we prove that $\lceil A,G \rceil$ is a filter of R if G is a filter in R .

1.4. Lemma : If A is any non-empty subset of an ADL R and G is a filter in $R.$ then $\lceil A,G \rceil$ is a filter of R and $G \subseteq \lceil A,G \rceil$

Proof : Let $R.$ be an ADL and A is any nonempty subset of $R.$ Let G be a filter in $R.$ Then for any $g \in G, a \vee g \in G,$ for all $a \in A.$ Therefore $G \subseteq \lceil A,G \rceil$.

Now $x, y \in \lceil A,G \rceil \Rightarrow a \vee x, \ a \vee y \in G,$ for all $a \in A$

$\Rightarrow (a \vee x) \wedge (a \vee y) \in G$ for all $a \in A$

$\Rightarrow a \vee (x \wedge y) \in G$ for all $a \in A$

$\Rightarrow x \wedge y \in \lceil A,G \rceil$.

Again, let $x \in \lceil A,G \rceil$. Then $a \vee x \in G$ for all $a \in A$.

Now, for any $r \in R$, $a \vee r \vee x = a \vee r \vee a \vee x \in G$ for all $a \in A$. Therefore $r \vee x \in \lceil A,G \rceil$.

Hence, $\lceil A,G \rceil$ is a filter in R.

Now, we introduce the concept of a relative dual annihilator in the following definition.

1.5. Definition : Let A be any non-empty subset of an ADL R. and G is a filter in R. Then we call the filter $\lceil A,G \rceil$ as a relative dual annihilator of A with respect to the filter G.

The following results can be proved easily.

1.6. Lemma : If A is a filter in R, then $\lceil A,A \rceil = R$.

1.7. Lemma : If B is a filter in an ADL R. and $A \subseteq B$ then $\lceil A,B \rceil = R$.

Proof : Let B be a filter in R. and A is any sub set of R such that $A \subseteq B$.

Then from (1) of Lemma 1.3, we get $\lceil A,A \rceil \subseteq \lceil A,B \rceil$. Therefore from Lemma 1.6,

we get $R \subseteq \lceil A,B \rceil$ and hence we get $\lceil A,B \rceil = R$.

1.8. Lemma : If $A = \{0\}$ and G is any filter in R, then $\lceil A,G \rceil = G$

1.9. Lemma : Let A, B be two non empty subsets of an ADL R and G is any filter of R.

Then 1). $\lceil A,G \rceil \subseteq A \cap \lceil B, \lceil B,G \rceil \rceil$

2). If $A \subseteq B$ then $\lceil A, \lceil B,G \rceil \rceil = \lceil A,G \rceil$

3). $\lceil A \cup B,G \rceil \subseteq \lceil A, \lceil B,G \rceil \rceil \subseteq \lceil A \cap B,G \rceil$

Proof : Let A, B be two non-empty subsets of an ADL R and G is a filter in R.

1). Since G is a filter in R, from Lemma 1.4, we have $G \subseteq \lceil B,G \rceil$ and hence from (1) of Lemma 1.3, we get $\lceil A,G \rceil \subseteq \lceil A, \lceil B,G \rceil \rceil \to$ (I)

Again from (2) of Lemma 1.3, we have $A \cap B \subseteq A$, $\lceil A, \lceil B,G \rceil \rceil \subseteq \lceil A \cap B, \lceil B,G \rceil \rceil \to$ (II)

Therefore, from (I) and (II), we get $\lceil A,G \rceil \subseteq \lceil A \cap B, \lceil B,G \rceil \rceil$

2). Let G be any filter in an ADL R. and A, B be two non-empty subsets of R such that $A \subseteq B$. Since $G \subseteq \lceil B,G \rceil$, we have $\lceil A,G \rceil \subseteq \lceil A, \lceil B,G \rceil \rceil \to$ (I)

Now, $x \in \lceil A, \lceil B,G \rceil \rceil \Rightarrow A \vee x \subseteq \lceil B,G \rceil$

$\Rightarrow a \vee x \in \lceil B,G \rceil$ for all $a \in A$

$\Rightarrow B \vee a \vee x \subseteq G$ for all $a \in A$

$\Rightarrow a \vee a \vee x \in G$ for all $a \in A (\subseteq B)$

$\Rightarrow a \vee x \in G$ for all $a \in A$

$\Rightarrow x \in \lceil A,G \rceil$

Therefore $\lceil A, \lceil B,G \rceil \rceil \subseteq \lceil A,G \rceil \to$ (II)

Thus from (I) and (II), we get $\lceil A, \lceil B,G \rceil \rceil = \lceil A,G \rceil$

3). Let A, B be two non-empty subsets of an ADL R and G is a filter in R.

Now, $A \subseteq A \cup B \Rightarrow \lceil A \cup B,G \rceil \subseteq \lceil A,G \rceil \to$ (I)

Again, $G \subseteq \lceil B,G \rceil \Rightarrow \lceil A,G \rceil \subseteq \lceil A, \lceil B,G \rceil \rceil \to$ (II)

Also, $A \cap B \subseteq A \Rightarrow \lceil A, \lceil B,G \rceil \rceil \subseteq \lceil A \cap B,G \rceil \to$ (III)

Therefore from (I), (II) and (III); we get $\lceil A\cup B,G\rceil\subseteq\lceil A,\lceil B,G\rceil\rceil\subseteq\lceil A\cap B,G\rceil$

Since $A\subseteq A$, from (2) of Lemma 1.9, we get the following corollary.

1.10. Corollary : If A is a non empty subset of an ADL R and G is any filter in R then $\lceil A,\lceil A,G\rceil\rceil=\lceil A,G\rceil$

1.11. Definition : For any $a,b\in R,$ define $\lceil a,b\rceil=\{x\in R \mid a\vee x=a\vee x\vee b\}.$

Observe that $x\in\lceil a,b\rceil \Leftrightarrow a\vee x=a\vee x\vee b \Leftrightarrow b=(a\vee x)\wedge b$

In the following result we prove that $\lceil a,b\rceil$ is a filter .

1.12. Theorem : For any $a,b\in R,$ $\lceil a,b\rceil=\lceil [a),[b)\rceil$

Proof : Let a,b be any two elements in $R.$

Now, $x\in\lceil [a),[b)\rceil \Rightarrow s\vee x\in[b),$ for all $s\in[a)$

$\Rightarrow a\vee x\in[b)$ (Since $a\in[a)$)

$\Rightarrow a\vee x=a\vee x\vee b \Rightarrow x\in\lceil a,b\rceil.$

Therefore $\lceil [a),[b)\rceil\subseteq\lceil a,b\rceil$

Let s be any element of $[a)$ .Then $s\vee a=s$.

Now, $x\in\lceil a,b\rceil \Rightarrow a\vee x\vee b=a\vee x$

$\Rightarrow a\vee x\in[b)$ (since $b\in[b)$)

$\Rightarrow s\vee a\vee x\in[b)$ (since $[b)$ is a filter)

$\Rightarrow s\vee x\in[b)$ (since $s\vee a=s$)

$\Rightarrow x\in\lceil [a),[b)\rceil$

Therefore $\lceil a,b\rceil\subseteq\lceil [a),[b)\rceil$. Thus $\lceil a,b\rceil=\lceil [a),[b)\rceil$.

1.13. Corollary : For any two elements a,b of an ADL R , $\lceil a,b\rceil$ is a filter.

Proof : For any $b\in R$, $[b)$ is a filter in R . Therefore from Lemma 1.4, we get $\lceil [a),[b)\rceil$ is a filter in R and hence from above Theorem 1.12, $\lceil a,b\rceil$ is a filter in R .

1.14. Theorem : Let R be an ADL and $a,b,c\in R$. Then

1). For any $a\in R$, $\lceil a,0\rceil=R$.

2). For any $a,b\in R,\lceil a,a\rceil=\lceil a,a\wedge b\rceil=\lceil a,b\wedge a\rceil=\lceil b\vee a,a\rceil=\lceil a\vee b,a\rceil=R.$

3). If $b\leq a$ then $\lceil a,b\rceil=R.$

4). For any $a,b,c\in R,\lceil a,b\wedge c\rceil=\lceil a,c\wedge b\rceil$ and $\lceil a,b\vee c\rceil=\lceil a,c\vee b\rceil.$

5). For any $a,b,c\in R,\lceil a,b\rceil\cap\lceil a,c\rceil\subseteq\lceil a,b\wedge c\rceil.$

6). If $b\leq c$ then $\lceil b,a\rceil\subseteq\lceil c,a\rceil$ and $\lceil a,c\rceil\subseteq\lceil a,b\rceil$

7). For any $a,b,c\in R.,$ $\lceil a,b\vee c\rceil=\lceil a,b\rceil\cap\lceil a,c\rceil.$

Proof : Let R be an ADL and $a,b,c\in R.$

1) is clear.

2). $\lceil a,a\rceil=R$ is clear.

Now $x\in\lceil a,a\wedge b\rceil \Leftrightarrow a\vee x=a\vee x\vee(a\wedge b)$

$\Leftrightarrow a\vee x=a\vee x\vee a\vee(a\wedge b)$ (since $a\vee b=a\vee b\vee a$)

$\Leftrightarrow a\vee x=a\vee x\vee a \Leftrightarrow x\in\lceil a,a\rceil$

Therefore $\lceil a,a\rceil=\lceil a,a\wedge b\rceil$

Similarly, we can prove $\lceil a, b\wedge a\rceil = \lceil b\vee a, a\rceil$ = $\lceil a\vee b, a\rceil = R.$

3). $b \le a \Rightarrow b\wedge a = a$. Hence the proof follows from (2)

4). Let a, b, c be any three elements of an ADL R .

Now, $x\in\lceil a, b\wedge c\rceil \Rightarrow a\vee x = a\vee x\vee (b\wedge c)$

$\Rightarrow b\wedge c = (a\vee x)\wedge (b\wedge c)$

$\Rightarrow (b\wedge c)\wedge (c\wedge b) = (a\vee x)\wedge (b\wedge c)\wedge (c\wedge b)$

$\Rightarrow c\wedge b = (a\vee x)\wedge (c\wedge b)$

$\Rightarrow a\vee x = a\vee x\vee (c\wedge b)$

$\Rightarrow x\in\lceil a, c\wedge b\rceil$

Therefore $\lceil a, b\wedge c\rceil \subseteq \lceil a, c\wedge b\rceil$.

Similarly, we get $\lceil a, c\wedge b\rceil \subseteq \lceil a, b\wedge c\rceil$

Hence we have $\lceil a, b\wedge c\rceil = \lceil a, c\wedge b\rceil$

Similarly, we can prove $\lceil a, b\vee c\rceil = \lceil a, c\vee b\rceil$

5). Let a, b, c be any three elements of an ADL R

Now $x\in\lceil a, b\rceil \cap \lceil a, c\rceil \Rightarrow x\in\lceil a, b\rceil$ and $x\in\lceil a, c\rceil$

$\Rightarrow a\vee x = a\vee x\vee b$ and $a\vee x = a\vee x\vee c$

$\Rightarrow b = (a\vee x)\wedge b$ and $c = (a\vee x)\wedge c$

$\Rightarrow b\wedge c = (a\vee x)\wedge b\wedge (a\vee x)\wedge c$

$\Rightarrow b\wedge c = (a\vee x)\wedge b\wedge c$

$\Rightarrow a\vee x = a\vee x\vee (b\wedge c)$

$\Rightarrow x\in\lceil a, b\wedge c\rceil$

Therefore $\lceil a, b\rceil \cap \lceil a, c\rceil \subseteq \lceil a, b\wedge c\rceil$

6). Let a, b, c be any three elements of an ADL R .

We have $b \le c \Rightarrow b\vee c = c = c\vee b$ and $b\wedge c = b$

Now, $x\in\lceil b, a\rceil \Rightarrow b\vee x = b\vee x\vee a$

$\Rightarrow c\vee b\vee x = c\vee b\vee x\vee a$

$\Rightarrow c\vee x = c\vee x\vee a$

$\Rightarrow x\in\lceil c, a\rceil$.

Therefore $\lceil b, a\rceil \subseteq \lceil c, a\rceil$

Again, $x\in\lceil a, c\rceil \Rightarrow a\vee x = a\vee x\vee c$

$\Rightarrow c = (a\vee x)\wedge c$

$\Rightarrow b\wedge c = b\wedge (a\vee x)\wedge c$

$\Rightarrow b\wedge c = (a\vee x)\wedge b\wedge c$

$\Rightarrow b = (a\vee x)\wedge b$

$\Rightarrow a\vee x = a\vee x\vee b$

$\Rightarrow x\in\lceil a, b\rceil$

Therefore $\lceil a, c\rceil \subseteq \lceil a, b\rceil$

7). Let a, b, c be any three elements of an ADL R

From(6), we have $b \le b\vee c \Rightarrow \lceil a, b\vee c\rceil \subseteq \lceil a, b\rceil$ and $c \le c\vee b \Rightarrow \lceil a, c\vee b\rceil \subseteq \lceil a, c\rceil$

Since $\lceil a, b\vee c\rceil$ = $\lceil a, c\vee b\rceil$, we get $\lceil a, b\vee c\rceil \subseteq \lceil a, b\rceil \cap \lceil a, c.\rceil$

Again, $x \in \lceil a,b \rceil \cap \lceil a,c \rceil \Rightarrow x \in \lceil a,b \rceil$ and $x \in \lceil a,c \rceil$

$\Rightarrow a \vee x = a \vee x \vee b$ and $a \vee x = a \vee x \vee c$

$\Rightarrow a \vee x = a \vee x \vee b \vee a \vee x \vee c$

$\Rightarrow a \vee x = a \vee x \vee b \vee c$

$\Rightarrow x \in \lceil a, b \vee c \rceil$

Therefore $\lceil a,b \rceil \cap \lceil a,c \rceil \subseteq \lceil a, b \vee c \rceil$.

Hence $\lceil a, b \vee c \rceil = \lceil a,b \rceil \cap \lceil a,c \rceil$.

1.15. Theorem : Let A be any non-empty subset of an ADL R and F be a filter in R, then $\lceil A, F \rceil = \bigcap_{a \in A} \lceil [a), F \rceil$.

Proof : Let F be any filter of R and A is a non-empty subset of R.

Now $x \in \bigcap_{a \in A} \lceil [a), F \rceil \Rightarrow x \in \lceil [a), F \rceil$ for all $a \in A$.

$\Rightarrow s \vee x \in F$ for all $s \in [a)$ and for all $a \in A$.

$\Rightarrow a \vee x \in F$ for all $a \in A$.

$\Rightarrow x \in \lceil A, F \rceil$

Therefore $\bigcap_{a \in A} \lceil [a), F \rceil \subseteq \lceil A, F \rceil$.

Let a be any element of A. Take $s \in [a)$. Then $s \vee a = s$.

Now $x \in \lceil A, F \rceil \Rightarrow a \vee x \in F$ for all $a \in A$.

$\Rightarrow s \vee a \vee x \in F$ for all $s \in [a)$ (since F is a filter) and for all $a \in A$

$\Rightarrow s \vee x \in F$ for all $s \in [a)$ and for all $a \in A$

$\Rightarrow [a) \vee x \subseteq$ F, for all $a \in A$

$\Rightarrow x \in \lceil [a), F \rceil$, for all $a \in A$

$\Rightarrow x \in \bigcap_{a \in A} \lceil [a), F \rceil$.

Therefore $\lceil A, F \rceil \subseteq \bigcap_{a \in A} \lceil [a), F \rceil$

Hence $\lceil A, F \rceil = \bigcap_{a \in A} \lceil [a), F \rceil$

1.16. Corollary : Let $x \in R$ and for any non-empty subset A of R,

$$\lceil A, [x) \rceil = \bigcap_{a \in A} \lceil a, x \rceil.$$

Let R be an ADL with maximal elements. Now, we recall that

$(s)^{+} = \{x \in R \mid s \vee x$ is a maximal element$\}$ is a filter in R.

Now we prove the following results.

1.17. Lemma : Let m_1, m_2 be two maximal elements in ADL R. Then for any $a \in R$,

$$\lceil a, m_1 \rceil = \lceil a, m_2 \rceil$$

Proof : Let m_1, m_2 be two maximal elements and a is any element of R.

Now, $x \in \lceil a, m_1 \rceil \Rightarrow a \vee x = a \vee x \vee m_1$

$\Rightarrow m_1 = (a \vee x) \wedge m_1$

$\Rightarrow m_1 \wedge m_2 = (a \vee x) \wedge m_1 \wedge m_2$

$\Rightarrow m_2 = (a \vee x) \wedge m_2$ (since m_1 is maximal)

$\Rightarrow a \vee x = a \vee x \vee m_2$

$\Rightarrow \quad x \in \lceil a, m_2 \rceil$

Therefore $\lceil a, m_1 \rceil \subseteq \lceil a, m_2 \rceil$.

Similarly, we can prove that $\lceil a, m_2 \rceil \subseteq \lceil a, m_1 \rceil$.

Thus, we get $\lceil a, m_1 \rceil = \lceil a, m_2 \rceil$.

1.18. Lemma : Let R be an ADL with maximal element m .
Then for any $a \in R,\ (a)^+ = \lceil a, m \rceil$

Proof : Let R be an ADL with maximal element m and $a \in R.$

Now, $x \in \lceil a, m \rceil \Rightarrow a \vee x = a \vee x \vee m$

$\Rightarrow a \vee x$ is maximal

$\Rightarrow x \in (a)^+$. Therefore $\lceil a, m \rceil \subseteq (a)^+$.

Let $x \in (a)^+ \Rightarrow a \vee x$ is maximal

$\Rightarrow a \vee x = m_1$ (say)

$\Rightarrow a \vee x = a \vee x \vee m_1 \Rightarrow x \in \lceil a, m_1 \rceil$

$\Rightarrow x \in \lceil a, m \rceil$ [from Lemma 1.17]

Therefore $(a)^+ \subseteq \lceil a, m \rceil$.

Thus we have $(a)^+ = \lceil a, m \rceil$

In the following theorem, we prove a relation between relative annihilators and relative dual annihilators in a relatively complemented ADL.

1.19. Theorem : Let R be a relatively complemented ADL with a maximal element m. Let $a, b \in R$ and a^m, b^m be the complements of a, b in $[0, a \vee m]$ and $[0, b \vee m]$ respectively. Then for any $x \in R,$ (1) $x \in \lceil a, b \rceil \Leftrightarrow x^m \in \lfloor a^m, b^m \rfloor$ and

(2) $x \in \lfloor a, b \rfloor \Leftrightarrow x^m \in \lceil a^m, b^m \rceil$

Proof : Let R be a relatively complemented ADL with maximal element m and a^m, b^m be the complements of a, b in $[0, a \vee m]$ and $[0, b \vee m]$ respectively.

(1) : Now $x \in \lceil a, b \rceil \Rightarrow a \vee x = a \vee x \vee b$

$\Rightarrow (a \vee x)^m = (a \vee x \vee b)^m$

$\Rightarrow a^m \wedge x^m = a^m \wedge x^m \wedge b^m$

$\Rightarrow a^m \wedge x^m \wedge a^m = a^m \wedge x^m \wedge b^m \wedge a^m$

$\Rightarrow x^m \wedge a^m = b^m \wedge x^m \wedge a^m$

$\Rightarrow x^m \in \lfloor a^m, b^m \rfloor$

Conversely, $x^m \in \lfloor a^m, b^m \rfloor \Rightarrow x^m \wedge a^m = b^m \wedge x^m \wedge a^m$

$\Rightarrow (x^m \wedge a^m)^m = (b^m \wedge x^m \wedge a^m)^m$

$\Rightarrow (x^m)^m \vee (a^m)^m = (b^m)^m \vee (x^m)^m \vee (a^m)^m$

$\Rightarrow (x \wedge m) \vee (a \wedge m) = (b \wedge m) \vee (x \wedge m) \vee (a \wedge m)$

$\Rightarrow b \wedge m = (b \wedge m) \wedge [(x \wedge m) \vee (a \wedge m)]$

$\Rightarrow b \wedge m = (b \wedge m) \wedge (x \vee a) \wedge m$

$\Rightarrow b \wedge m = (a \vee x) \wedge (m \wedge b) \wedge m$

$\Rightarrow b \wedge m \wedge b = (a \vee x) \wedge b \wedge m \wedge b$

$\Rightarrow$ b $= (a \vee x) \wedge b$

$$\Rightarrow \quad a \vee x = a \vee x \vee b$$
$$\Rightarrow \quad x \in \lceil a,b \rceil$$

Therefore $x \in \lceil a,b \rceil \Leftrightarrow x^m \in \lfloor a^m, b^m \rfloor$.

2) : Now $x \in \lfloor a,b \rfloor \Rightarrow x \wedge a = b \wedge x \wedge a$

$$\Rightarrow (x \wedge a)^m = (b \wedge x \wedge a)^m$$
$$\Rightarrow x^m \vee a^m = b^m \vee x^m \vee a^m$$
$$\Rightarrow b^m = b^m \wedge (x^m \vee a^m)$$
$$\Rightarrow b^m \wedge b^m = b^m \wedge (x^m \vee a^m) \wedge b^m$$
$$\Rightarrow b^m = (a^m \vee x^m) \wedge b^m$$
$$\Rightarrow a^m \vee x^m = a^m \vee x^m \vee b^m$$
$$\Rightarrow x^m \in \lceil a^m, b^m \rceil$$

Conversely, $x^m \in \lceil a^m, b^m \rceil \Rightarrow a^m \vee x^m = a^m \vee x^m \vee b^m$

$$\Rightarrow (a \wedge x)^m = (a \wedge x \wedge b)^m$$
$$\Rightarrow a \wedge x \wedge m = a \wedge x \wedge b \wedge m$$
$$\Rightarrow a \wedge x \wedge m \wedge a = a \wedge x \wedge b \wedge m \wedge a$$
$$\Rightarrow x \wedge a = b \wedge x \wedge a$$
$$\Rightarrow x \in \lfloor a,b \rfloor$$

Therefore $x \in \lfloor a,b \rfloor \Leftrightarrow x^m \in \lceil a^m, b^m \rceil$

2. Relatively dually normal ADLs

Now we introduce the concept of a relatively dually normal ADL in a natural way by defining every closed interval in R is dually normal.

In this section the letter R stands for an ADL with maximal elements.

2.1. Definition : An ADL R is said to be relatively dually normal, if for any $a, b \in R$ with $a < b$, the interval $[a,b]$ is dually normal.

2.2. Definition : Let x be any element of an ADL R and S is any subset of R .

Then we define $(x)_S^+ = \{s \in S \mid x \vee s$ is maximal in S}

The following lemma can be proved easily using the above definition.

2.3. Lemma : Let a, b be any two elements of an ADL R with $a < b$.

Then for $x \in [a,b]$, $(x)_{[a,b]}^+ = \lceil x,b \rceil \cap [a,b]$ and $(x)_{[a,b]}^+$ is a filter in the interval $[a,b]$, Now we prove the following Lemma.

2.4. Lemma : Let R be an ADL and with $a < b$. Then for any $x, y \in [a,b]$, $\{\lceil x,b \rceil \vee \lceil y,b \rceil\} \cap [a,b] = \{\lceil x,b \rceil \cap [a,b]\} \vee \{\lceil y,b \rceil \cap [a,b]\}$

Proof : Let R be an ADL and $a, b \in R$ with $a < b$ and $x, y \in [a,b]$.

Now, $\lceil x,b \rceil \subseteq \lceil x,b \rceil \vee \lceil y,b \rceil \Rightarrow \lceil x,b \rceil \cap [a,b] \subseteq \{\lceil x,b \rceil \vee \lceil y,b \rceil\} \cap [a,b]$. Similarly, $\lceil y,b \rceil \subseteq \lceil x,b \rceil \vee \lceil y,b \rceil \Rightarrow \lceil y,b \rceil \cap [a,b] \subseteq \lceil x,b \rceil \vee \lceil y,b \rceil \cap [a,b]$ Therefore $\{\lceil x,b \rceil \cap [a,b]\} \vee \{\lceil y,b \rceil \cap [a,b]\} \subseteq \{\lceil x,b \rceil \vee \lceil y,b \rceil\} \cap [a,b] \rightarrow$ (I)

Now, $p \in \{\lceil x,b \rceil \vee \lceil y,b \rceil\} \cap [a,b] \Rightarrow p \in \{\lceil x,b \rceil \vee \lceil y,b \rceil\}$ and $p \in [a,b]$

$$\Rightarrow p = s_1 \wedge s_2 \quad \text{where} \quad s_1 \in \lceil x,b \rceil, \quad s_2 \in \lceil y,b \rceil$$

$\Rightarrow\ p = s_1 \wedge s_2$ where $x \vee s_1 = x \vee s_1 \vee b,\ \ y \vee s_2 = y \vee s_2 \vee b$

$\Rightarrow\ p = s_1 \wedge s_2$ where $b = (x \vee s_1) \wedge b,\ \ b = (y \vee s_2) \wedge b$

Also, $p \in [a,b] \ \Rightarrow\ p = (a \vee p) \wedge b$

$\Rightarrow\ p = [a \vee (s_1 \wedge s_2)] \wedge b$ (since $p = s_1 \wedge s_2 \in [a,b]$)

$\Rightarrow\ p = (a \vee s_1) \wedge (a \vee s_2) \wedge b$

$\Rightarrow\ p = [(a \vee s_1) \wedge b] \wedge [(a \vee s_2) \wedge b]$

Clearly, $(a \vee s_1) \wedge b \in [a,b]$ and $(a \vee s_2) \wedge b \in [a,b]$

Now we prove that $(a \vee s_1) \wedge b \in \lceil x,b \rceil$ and $(a \vee s_2) \wedge b \in \lceil y,b \rceil$.

Now, $\{x \vee [(a \vee s_1) \wedge b]\} \wedge b \ = \ (x \vee a \vee s_1) \wedge (x \vee b) \wedge b$

$= \ (a \vee x \vee s_1) \wedge b$

$= \ (x \vee s_1) \wedge b$ (since $x \in [a,b],\ \ a \vee x = x$)

$= \ (x \vee s_1 \vee b) \wedge b$ (since $x \vee s_1 = x \vee s_1 \vee b$)

$= \ b$

This gives $x \vee [(a \vee s_1) \wedge b] \vee b \ = \ x \vee [(a \vee s_1) \wedge b]$.

Therefore $(a \vee s_1) \wedge b \in \lceil x,b \rceil$ and hence $(a \vee s_1) \wedge b \ \in \ \lceil x,b \rceil \cap [a,b]$.

Similarly, we get $(a \vee s_2) \wedge b \in \lceil y,b \rceil \cap [a,b]$

Therefore $p \ = \ [(a \vee s_1) \wedge b] \ \wedge \ [(a \vee s_2) \wedge b] \ \in \ \{\lceil x,b \rceil \cap [a,b]\} \ \vee \ \{\lceil y,b \rceil \cap [a,b]\}$.

and hence $\{\lceil x,b \rceil \vee \lceil y,b \rceil\} \cap [a,\ b] \ \subseteq \ \{\lceil x,b \rceil \cap [a,b]\} \ \vee \ \{\lceil y,b \rceil \cap [a,b]\} \ \rightarrow (II)$

Therefore from (I) and (II), we get

$$\{\lceil x,b \rceil \vee \lceil y,b \rceil\} \cap [a,\ b] \ = \ \{\lceil x,b \rceil \cap [a,b]\} \ \vee \ \{\lceil y,b \rceil \cap [a,b]\}$$

In the following theorem, we characterize a relatively dually normal ADL in terms of relative dual annihilators.

2.5. Theorem : Let R be an ADL with maximal elements. Then the following are equivalent.

1). R is relatively dually normal.

2). $R \ = \ \lceil a,b \rceil \vee \lceil b,a \rceil$ for all $a,b \in R$

3). $\lceil a \vee b,c \rceil = \lceil a,c \rceil \ \vee \ \lceil b,c \rceil$ for all $a,b,c \in R$

Proof :

(1) $\Rightarrow$ (2) : Assume that R is a relatively dually normal ADL.

Let $a,b \in R$. Then clearly, $\lceil a,b \rceil \vee \lceil b,a \rceil \ \subseteq \ R$.

Let R $c \in R$. Now consider the interval $I = [c, c \vee a \vee b]$. Then I is dually normal.

Since $c \vee a \vee b = c \vee a \vee c \vee b$ is maximal in I, from Theorem 0.5, there exist $a_1, b_1 \in I$ such that $c \vee a \vee a_1,\ c \vee b \vee b_1$ are maximal elements in I and $a_1 \wedge b_1 = c$ (the zero element of I).

Clearly, $c \vee a \vee a_1 = c \vee a \vee b \ = \ c \vee b \vee b_1$ and $c = c \vee (a_1 \wedge b_1) \ = \ (c \vee a_1) \wedge (c \vee b_1)$.

Now, we prove that $c \vee a_1 \in \lceil a,b \rceil$ and $c \vee b_1 \in \lceil b,a \rceil$

Now, $a \vee c \vee a_1 \vee b \ = \ a \vee c \vee a_1 \vee c \vee a \vee b$

$= \ a \vee c \vee a_1 \vee c \vee a \vee a_1$

$= \ a \vee c \vee a_1$

Therefore, $c \vee a_1 \in \lceil a,b \rceil$. Similarly, we can prove that $c \vee b_1 \in \lceil b,a \rceil$

Therefore, $c \ = \ (c \vee a_1) \wedge (c \vee b_1) \ \in \ \lceil a,b \rceil \vee \lceil b,a \rceil$

Thus we get $R \ \subseteq \ \lceil a,b \rceil \vee \lceil b,a \rceil$. Hence $R \ = \ \lceil a,b \rceil \vee \lceil b,a \rceil$

(2) $\Rightarrow$ (3) : Assume the condition (2). Let $s \in S$ $a,b,c \in R$

From Theorem 1.13, we have $\lceil a,c \rceil \vee \lceil b,c \rceil \subseteq \lceil a \vee b,c \rceil$

Let $x \in \lceil a \vee b,c \rceil$. Then $a \vee b \vee x = a \vee b \vee x \vee c$

Since $x \in R$, we can write $x = s_1 \wedge s_2$ where $s_1 \in \lceil a,b \rceil$, $s_2 \in \lceil b,a \rceil$

That is, $x = s_1 \wedge s_2$ and $a \vee s_1 = a \vee s_1 \vee b$, $b \vee s_2 = b \vee s_2 \vee a$

Now we prove that $s_1 \in \lceil a,c \rceil$ and $s_2 \in \lceil b,c \rceil$

Since $x = s_1 \wedge s_2$, we have $s_1 = s_1 \vee (s_1 \wedge s_2) = s_1 \vee x$

Now, $a \vee s_1 = a \vee s_1 \vee x$ (since $s_1 = s_1 \vee x$)

$= a \vee s_1 \vee b \vee x$ (since $a \vee s_1 = a \vee s_1 \vee b$)

$= a \vee s_1 \vee a \vee b \vee x$

$= a \vee s_1 \vee a \vee b \vee x \vee c$ (since $a \vee b \vee x = a \vee b \vee x \vee c$)

$= a \vee s_1 \vee b \vee x \vee c$

$= a \vee s_1 \vee b \vee s_1 \vee x \vee c$

$= a \vee s_1 \vee b \vee s_1 \vee c$ (since $s_1 \vee x = s_1$)

$= a \vee s_1 \vee b \vee c$

$= a \vee s_1 \vee c$ (since $a \vee s_1 \vee b = a \vee s_1$)

Therefore $s_1 \in \lceil a,c \rceil$. Similarly, we can prove that $s_2 \in \lceil b,c \rceil$

Therefore $x = s_1 \wedge s_2 \in \lceil a,c \rceil \vee \lceil b,c \rceil$ and hence $\lceil a \vee b,c \rceil \subseteq \lceil a,c \rceil \vee \lceil b,c \rceil$

Therefore $\lceil a \vee b,c \rceil = \lceil a,c \rceil \vee \lceil b,c \rceil$ for all $a,b,c \in R$

(3) $\Rightarrow$ (1) : Assume the given condition (3). We have to prove that R is relatively dually normal. Let $a,b \in R$ and $a < b$.

Consider the interval $[a,b]$ and $x, y \in [a,b]$. Then from Theorem 0.6, it is enough to prove that $(x \vee y)^+_{[a,b]} = (x)^+_{[a,b]} \vee (y)^+_{[a,b]}$

Now, $(x \vee y)^+_{[a,b]} = \lceil x \vee y,b \rceil \cap [a,b]$ from Lemma 2.3.

$= \{\lceil x,b \rceil \vee \lceil y,b \rceil\} \cap [a,b]$

$= \{\lceil x,b \rceil \cap [a,b]\} \vee \{\lceil y,b \rceil \cap [a,b]\}$ from Lemma 2.4.

$= (x)^+_{[a,b]} \vee (y)^+_{[a,b]}$

Therefore $[a,b]$ is dually normal and hence R is relatively dually normal.

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