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THE IMPROVEMENT OF MODIFIED NEWTON'S METHOD

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Abstract

A new variant of Newton's method based on mid- point method has been developed and their convergence properties have been discussed. The order of convergence of the proposed method is five. Starting with a suitably chosen x_0 , the method generates a sequence of iterates converging to the root. The convergence analysis is provided to establish its fifth order of convergence. It does not require the evaluation of the second order derivative of the given function as required in the family of Chebyshev–Halley type methods. Analysis of efficiency shows that the new method can compete with Newton's method and the classical third order methods. Numerical results show that the method has definite practical utility.

Keywords:

Newton's method;
Iteration function;
Order of convergence;
Function evaluations;
Efficiency index.

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1. Introduction

In Science and Engineering, many of the nonlinear and transcendental problems, of the form $f(x) = 0$, are complex in nature. Since it is not always possible to obtain its exact solution by usual algebraic process, therefore numerical iterative methods such as Newton, secant methods are often used to obtain the approximate solution of such problems. These methods are very effective, but there are some limitations that they do not give the result as fast as we want, and take several iterations. There are so many methods developed on the improvement of quadratically convergent Newton's method so as to get a superior convergence order than Newton.

This paper is concerned with the iterative methods for finding a simple root α , i.e. $f(\alpha) = 0$, and $f'(\alpha) \neq 0$ of $f(x) = 0$, where $f : R \to R$, be the continuously differentiable real function. Symbols have usual meanings.

We consider the problem of finding a real zero of a function f: $I \subset R \to R$. This zero can be determined as a fixed point of some iteration function g by means of the one-point iteration method

$$x_{n+1} = g(x_n), \qquad n = 0, 1, \cdots$$

where x_0 is the starting value, The best known and the most widely used example of these types of methods is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, \cdots \tag{1}$$

It converges quadratically to simple zeros and linearly to multiple zeros. In the literature, its several modifications have been introduced in order to accelerate it or to get a method with a higher order of

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convergence at the expense of additional evaluations of functions, derivatives and changes in the points of iterations.

If we consider the definition of *efficiency index* as $p^{1/m}$ where p is the order of the method and m is the number of functions evaluations required by the method (units of work per iteration) then the efficiency index of this method is 1.414. A number of ways are considered by many researchers to improve the local order of convergence of Newton's method at the expense of additional evaluations of functions, derivatives and changes in the points of iterations.

All these modifications are in the direction of increasing the local order of convergence with the view of increasing their efficiency indices. For example, the method developed by Fernando et al. [1] called as trapezoidal Newton's or arithmetic mean Newton's method, suggests for some other variants of Newton's method. Frontini and Sormani [6] developed new modifications of Newton's method to produce iterative methods with order of convergence of three and efficiency index of 1.442. With the same efficiency index, A.Y. Ozban [2], Some new variants of Newton's method, Applied Mathematics Letters 17, (2004), 677- 682 and Traub [9] developed a third order method requiring evaluations of one function and two first derivatives per iteration. Chen [7] described some new iterative formulae having third order convergence.

2. Definitions

Definition 1: See [8, 9] Considering the problem of numerical approximation of a real roots α of the non linear equation:

$$f(x) = 0, \quad f: D \subseteq R \to R.$$

The root α is said to be simple if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. If $f(\alpha) = f'(\alpha) = \cdots = f^{m-1}(\alpha) = 0$ and $f^{m}(\alpha) \neq 0$ for $m \geq 1$ then α is of multiplicity m.

Definition 1: See [8, 9] If the sequence $\{ x_n / n \geq 0\}$ tends to a limit α in such a way that

$$\lim_{x_n \to \alpha} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = C = |g^{(p)}(\alpha)| / p! \quad (2)$$

For some $C \neq 0$ and $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as *asymptotic error constant*.

When $p = 1$, the convergence is *linear*, and it is called *first order convergence.*

While for $p = 2$ and $p = 3$ the sequence is said to *converge quadratically* and *cubically* respectively.

The value of p is called the *order of convergence* of the method which produces the sequence $\{ x_n : n \geq 0\}$. Let $e_n = x_n - \alpha$ Then the relation $e_{n+1} = C\, e_n^{\,p} + O(e_n^{\,p+1})$ is called the error equation for the method, p being the order of convergence.

Definition 3: See [8, 9] *Efficiency index* is simply defined as $p^{1/m}$ where p is the order of the method and m is the number of functions evaluations required by the method (units of work per iteration). Therefore the efficiency index of Newton´s method is 1.414 and that of third order method which take three functions evaluations is 1.442 and efficiency index of method proposed by me is 1.495.

3. Description of the methods

Let α be a simple zero of a sufficiently differentiable function *f* and consider the numerical solution of the equation $f(x) = O$. It is clear that – (See 1)

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \quad (3)$$

Suppose we interpolate f' on the interval $[x_n, x]$ by the constant $f'(x_n)$, then $(x - x_n) f'(x_n)$ provides an approximation for the integral in (2) and by taking $x = \alpha$ we obtain

$$0 \approx f(x_n) + (\alpha - x_n) f'(x_n)$$

and hence, a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, \cdots.$$

which is Newton's method for $n = 0, 1, \cdots$. On the other hand, if we approximate the integral in (2) by the trapezoidal rule and take x =α, we obtain

$0 \approx f(x_n) + 1/2\,(\alpha - x_n)(f'(x_n) + f'(\alpha))$

Therefore, a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(x_{n+1})}. \tag{4}$$

If the $(n+1)^{th}$ value of Newton's method is used on the right-hand side of the above equation to overcome the implicity problem, then

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(z_{n+1})}, \quad \text{where } z_{n+1} = x_n - f(x_n)/f'(x_n) \tag{5}$$

is obtained which is, for $n = 0, 1, \cdots$, the trapezoidal Newton's method of Fernando et al. [1]. Let us rewrite equation (3) as

$$x_{n+1} = x_n - \frac{f(x_n)}{(f'(x_n)+f'(z_{n+1}))/2}, \quad n = 0, 1, \cdots \tag{6}$$

So, this variant of Newton's method can be viewed as obtained by using arithmetic mean of $f'(x_n)$ and $f'(z_{n+1})$ instead of $f'(x_n)$ in Newton's method defined by (1) which was called as *arithmetic mean Newton's method.* On the other hand, if we approximate the integral in (2) by the midpoint integration rule instead of the trapezoidal rule and take $x = \alpha$, we obtain

$0 \approx f(x_n) + (\alpha - x_n)\, f'((x_n + \alpha)/2),$

and in this case a new approximation x_{n+1} to α is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f((x_n+z_{n+1})/2)}, \quad n = 0, 1, \cdots \tag{7}$$

Again using the $(n+1)^{th}$ value of Newton's method on the right-hand side of the last equation to overcome the implicity problem, we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f((x_n+z_{n+1})/2}, \quad \text{where} \quad z_{n+1} = x_n - f(x_n)/f'(x_n), \; n = 0, 1, \cdots \tag{8}$$

which is called midpoint *Newton's* (MN) *method.*

3.1 *New Variant of Newton´s Method*

Proposed method will be

$$y_n = x_n - f(x_n)/f'(x_n),$$

$$z_n = x_n - \frac{f(x_n)}{f((x_n+y_n)/2)} \tag{9}$$

$$x_{n+1} = z_n - f(z_n)/f'(z_n).$$

Now using the linear interpolation on two points $(x_n, f'(x_n))$ and we get $(y_n, f'(y_n))$, we get

$$f'(x) \approx \frac{x-x_n}{y_n-x_n} f'(y_n) + \frac{x-y_n}{x_n-y_n} f'(x_n) \tag{10}$$

Thus an approximation to $f'(z_n)$ is given by

$$f'(z_n) \approx \frac{z_n-x_n}{y_n-x_n} f'(y_n) + \frac{z_n-y_n}{x_n-y_n} f'(x_n)$$

Putting the values in the above equation and solving

$$f'(z_n) = \frac{3f'(x_n) \;\; f'((x_n+x_{n+1})/2) \; - 2f'(x_n)^2}{f'((x_n+x_{n+1})/2)}. \tag{11}$$

Hence

$$x_{n+1} = z_n - \frac{f(z_n)\ f'(\ (x_n+x_{n+1})/2\ )}{3\,f\,'(x_n)\quad f'(\ (x_n+x_{n+1})/2\ )\ -2\,f'(x_n)^2}. \tag{12}$$

This is proposed fifth order mid- point method.

4. Convergence Analysis

Theorem 1. - Let α ε I be a simple zero of a sufficiently differentiable function *f:* I ⊂ R → R for an open interval I. If x_0 is sufficiently close to α, then the methods defined by (12) has fifth order convergence.

*Proof***.** Let α ε I be a simple zero of *f*. Since *f* is sufficiently differentiable, by expanding $f\,(x_n)$ and $f'(x_n)$ about α we get

$$f\,(x_n) = f\,(\alpha+e_n) = f'(\alpha)e_n + \frac{1}{!2} f\,''(\alpha)\,e_n^2 + \frac{1}{!3} f'''(\alpha)\,e_n^3 + O\,(e_n^4)$$

$$f(x_n) = f'(\alpha)\,[\,e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)],\ \text{where}\ \ C_J = (1/!\,j) f^j\,(\alpha)/f'\,(\alpha)$$

$$f'(x_n) = f'(\alpha+e_n) = f'(\alpha) + f\,''(\alpha)e_n + \frac{1}{!3} f'''(\alpha)e_n^2 + O\,(e_n^4)$$

$$= f'(\alpha)\,[1+2C_2\,e_n + 3C_3 e_n^2 + 4C_4\,e_n^3 + O(e_n^4)] \tag{13}$$

$$\frac{f(x_n)}{f'(x_n)} = [\,e_n + C_2 e_n^2 + C_3\ e_n^3 + O\,(e_n^4)][1 + 2C_2\,e_n + 3\,C_3 e_n^2 + 4C_4\,e_n^3 + O\,(e_n^4)]^{-1}$$

$$= [e_n - C_2 e_n^2 + (2C_2^2 - 2\,C_3)\,e_n^3 + O(e_n^4)].$$

Newton method is

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$y_n = \alpha + C_2 e_n^2 + (2\,C_3 - 2C_2^2)\,e_n^3 + O\,(e_n^4) \tag{14}$$

Newton´s method converges quadratically. Hence,

$$x_{n+1} = z_n - \frac{f(z_n)\ f'((x_n+y_n)/2\ )}{3\,f'(x_n)\quad f'(\,(x_n+y_n)/2\ )\ -2\,f'(x_n)^2}\ . \tag{15}$$

Since

$$f'\left(\frac{x_n+y_n}{2}\right) = f'(\alpha) + [\tfrac{1}{2} e_n + \tfrac{1}{2}\{C_2\,e_n^2 + (2C_3 - 2C_2^2)e_n^3\}]f''(\alpha) + \tfrac{1}{8}\,\{e_n^2 + 2C_2^2\,e_n^3\}f'''(\alpha) + \tfrac{1}{48}\,e_n^3 f'''(\alpha) + O(e_n^4)],$$

$$f'(x_n) = f'(\alpha+e_n) = f'(\alpha) + f''(\alpha)e_n + f'''(\alpha)\,\frac{1}{!3}\,e_n^2 + O(e_n^4),$$

$$f'(x_n)^2 = f'(\alpha)^2[1+4C_2\,e_n + (4C_2^2 + 6C_3)e_n^2 + (8C_4 + 12C_2\,C_3)e_n^3 + O(e_n^4)].$$

Now

$$f(z_n) = f'(\alpha)\,[\,(\,C_2^2 - \tfrac{1}{4}C_3)e_n^3 + O(e_n^4)\} + \left(\,C_2^2 - \tfrac{1}{4}C^3\right)^2 e_n^6 + O(e_n^7)\,]$$

Putting all these values in equation – 15 and solving we get

$$e_{n+1} = -\frac{3}{2}\,C_3\,A\,e_n^5 + O(e_n^6). \tag{16}$$

Hence the proposed method defined by (12) has fifth order convergence.

5. Numerical results and discussion

Example 1: Consider the equation $x^3+4x^2-10 = 0$.
We start with initial approximations $x_0 = -0.5$ and 1. The results obtained by Newton iteration and present iteration are shown in Table 1.

Table 1. Comparison of present method with Newton's method

Mathod	x_0	n	x_n	$f(x_n)$
Newton iteration	0.5	126	1.795421944101975	8.681774206590596
		127	1.434193024269764	1.177643967594719
		128	1.367449330052116	0.036688346489555
		129	1.365232424149671	3.980948565818210e-005
		130	1.365230013416946	4.704858724835503e-011
		131	1.365230013414097	0.000000000000
Proposed formula	0.5	5	1.365230013413797	-4.945377440890297e-012
		6	1.365230013414097	0.000000000000
Newton iteration	1.0	1	1.454545454545455	1.540195341848236
		2	1.368900401069519	0.060719688639942
		3	1.365236600202116	1.087706104243580e-004
		4	1.365230013435367	3.512390378546115e-010
		5	1.365230013414097	0.000000000000
Proposed formula	1.0	1	1.359763010706868	-0.090036995907809
		2	1.365230013119094	-4.871500536296480e-009
		3	1.365230013414097	0.000000000000000

Example 2: Consider the equation $x^3 - e^{-x} = 0$.
We start with initial approximations $x_0 = 1$. The results obtained by Newton iteration and present iteration are shown in Table 2.

Table 2. Comparison of present method with Newton's method

Mathod	x_0	n	x_n	$f(x_n)$
Newton iteration	1.0	1	0.812309030097381	0.092166771534313
		2	0.774276548985500	0.003144824978613
		3	0.772884756209622	4.050085547491200e-006
		4	0.772882959152202	6.742550962002269e-012
		5	0.772882959149210	0.000000000000000
Proposed formula	1.0	1	0.772653567126747	-5.168765374116702e-004
		2	0.772882959149210	0.000000000000

Example 3: Consider the equation sin (x) - 0.5 $x = 0$.
We start with initial approximations $x_0 = 1.6$. The results obtained by Newton iteration and present iteration are shown in Table 2.

Faster convergence of proposed method for example 2 and example 3 are shown in Figure 1 and Figure 2 respectively. The line with red marks shows the proposed method and the line with black mark shows the Newton method in the figures.

Table 3. Comparison of present method with Newton's method

Mathod	x_0	n	x_n	$f(x_n)$

Newton iteration	1.6	1	1.977123551007066	-0.069983138933437
		2	1.898950910895084	-0.002836729003200
		3	1.895501147295299	-5.635111423818451e-006
		4	1.895494267061370	-2.243205621255129e-011
		5	1.895494267033981	0.000000000000000
Proposed formula	1.6	1	1.895612470811224	-9.681817693751871e-005
		2	1.895494267033981	0.000000000000

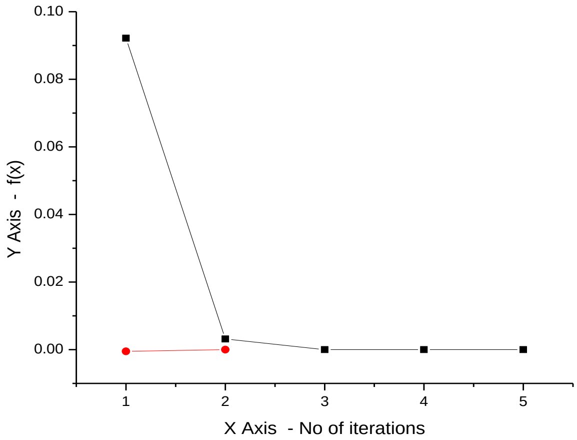

Figure 1.

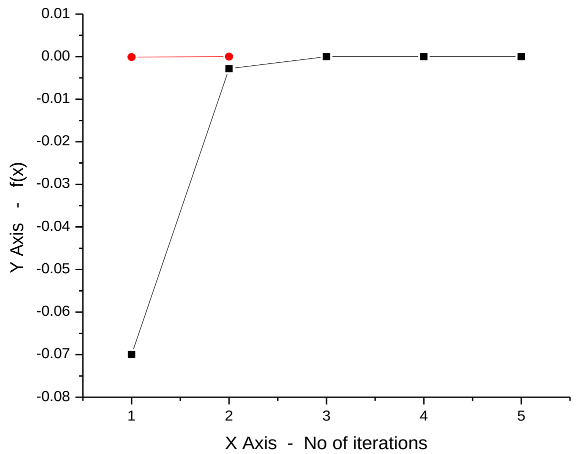

Figure 2.

In Table 4, we give the number of iterations (N) and the number of function evaluations (NOFE) required satisfying the stopping criterion, F denotes that method fails and D denotes for divergence. PM denotes proposed method. CN denotes for classical Newton method, AN-Arithmetic mean Newton method, HN-Harmonic mean Newton method, GN- Geometric mean Newton method, MN – Mid- point Newton method. In Table 4 the following test functions have been used:

(a) $x^3 + 4x^2$ - I0, α = 1.365230013414097,
(b) $\sin^2 x - x^2 + 1$, α = - 1.404491648215341,
(c) Cos $x - x$, α = 0.7390851332151607,
(d) x^3-10 α = 2.154434690031884,
(e) $x^3 - e^{-x}$ α = 0.772882959149210,
(f) $(x- 1)^3 - 1$, α = 2.

Hence from the table 1,2,3,4 the main observations are as follows:

I. Present method takes lesser number of iterations than the others compared here. [Table 1,2,3,4]

II. Examples show that the present method requires lesser number of function evaluations, as compared to other methods [Table 1,2,3,4].

Thus, the present method is not only faster but the cost effecting parameters obtain in examples shows that it has minimum cost among all the methods taken here.

Table 4 - Comparison with third order methods

F(x)	x_0	N						NOFE					
		CN	AN	HN	GN	MN	PM	CN	AN	HN	GN	MN	PM
(a)	-05	131	6	65	F	10	6	262	18	195	-	30	24
	1.5	4	3	3	3	3	2	8	9	9	9	9	8
(b)	1.9	5	3	3	3	3	2	10	9	9	9	9	8
(c)	3	6	9	5	F	4	3	12	27	15	-	12	12
(d)	-3	18	D	17	D	D	5	36	-	51	-	-	20
	2	4	3	3	D	3	2	8	9	9	-	9	8
(e)	0.5	5	4	3	13	3	2	10	12	9	39	9	8
	1	3	3	3	15	3	2	10	9	9	45	9	8
(f)	1.8	5	3	3	28	3	2	10	9	9	84	9	8
	3.5	7	5	4	37	5	3	14	15	12	111	15	12

6. Conclusions

We presented the results of some numerical tests to compare the efficiencies of the proposed method. We employed - CN method, some third order methods- AN method of Fernando et al. [1], HN, GN, MN methods. Numerical computations reported here have been carried out in MATLAB. The stopping criterion has been taken as $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-14}$.

We, proposed a fifth order method for finding simple real roots of nonlinear equations, in a new way, which is free from second order derivative of the given function as required in the family of Chebyshev–Halley type methods. Our method requires evaluations of two functions and two first order derivatives per iteration. The convergence analysis of the method is performed in much simpler way to show that the order of convergence of the method is five. The high order convergence is also corroborated by numerical tests.

Method has the efficiency index equal to 1.4953 which is better to Newton's method with efficiency index equal to 1.414 and the classical third order methods (1.442), such as Weerakoon and Fernando method, Chebyshev's method, Halley's method and Super-Halley method, fifth order method (1.495) of Kou, Li and Wang [13]. The method is tested on a number of numerical examples. On comparing our results with those obtained by Newton's method (NM) third order methods, it is found that our method is most effective as it converges to the root much faster.

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