OBSERVATIONS ON THE HOMOGENEOUS QUINTIC EQUATION WITH FOUR UNKNOWNS

$$x^5 - y^5 = 2z^5 + 5(x+y)(x^2 - y^2)w^2$$

S.Vidhyalakshmi*

K.Lakshmi*

M.A.Gopalan*

ABSTRACT

We obtain infinitely many non-zero integer quadruples (x, y, z, w) *satisfying the quintic equation with four unknowns.* $x^5 - y^5 = 2z^5 + 5(x+y)(x^2 - y^2)w^2$.*Various interesting properties among the values of x, y, z and w are presented.*

KEYWORDS: Quintic equation with four unknowns, integral solutions.

MSC 2000 Mathematics subject classification: 11D41.

NOTATIONS:

. $T_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right)$ -Polygonal number of rank n with size m

$P_n^m = \frac{1}{6}\ n(n+1)((m-2)n + (5-n))$ - Pyramidal number of rank n with size m

$PR_p = n(n+1)$-Pronic number of rank n

$S_n = 6n(n-1) + 1$-Star number of rank n

$J_n = \frac{1}{3}\ 2^n - (-1)^n$ -Jacobsthal number of rank n

$j_n = 2^n + (-1)^n$- Jacobsthal-Lucas number of rank n

$KY_n = (2^n + 1)^2 - 2$ -keynea number.

* Department of Mathematics, Shrimati Indira Gandhi College, Trichy

1. INTRODUCTION

The theory of diophantine equations offers a rich variety of fascinating problems. In particular,quintic equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-3].For illustration, one may refer [4-5] for quintic equation with three unknowns and [6-7] for quintic equation with five unknowns. This paper concerns with the problem of determining non-trivial integral solutions of the homogeneous quintic equation with four unknowns given by $x^5 - y^5 = 2z^5 + 5(x+y)(x^2-y^2)w^2$.A few relations among the solutions are presented.

2. Method of Analysis:

The diophantine equation representing the quintic equation with four unknowns under consideration is

$$x^5 - y^5 = 2z^5 + 5(x+y)(x^2-y^2)w^2 \qquad (1)$$

It is observed that (1) is satisfied by the following non-zero distinct integer quadruples:

$(x, y, z, w):(6k,-2k,4k,3k),(2(2k^2+2k-1),2(2k^2-2k-1),4k,2k^2+1)$

However, we have other patterns of solutions which are illustrated below:

2.1: Pattern I:

Introduction of the transformations

$$x = u+v, y = u-v, z = v \qquad (2)$$

in (1) leads to

$$u^2 + 2v^2 = 4.w^2 \qquad (3)$$

which is of the form $z^2 = Dx^2 + y^2$.

Using the most sited solution of the above equation, the corresponding non-zero distinct integral solutions of (1) are given by

$$\left.\begin{aligned} x &= 2p^2 - 4q^2 + 4pq \\ y &= 2p^2 - 4q^2 - 4pq \\ z &= 4pq \\ w &= p^2 + 2q^2 \end{aligned}\right\} \qquad (4)$$

Following are some interesting relations between the solutions of (1):

1. $x(p,p)+y(p,p)+z(p,p)+w(p,p)=2T_{5,p}+PR_p-T_{4,p}$

2. Each of the following expression is a nasty number:

(a) $6[4w(p,q)-x(p,q)-y(p,q)]$.

(b). $6[x(p,1)-y(p,1)-z(p,1)+w(p,1)]+12$

3. $x(p,q)-z(p,q)-w(p,q)-T_{4,p}+S_p-2T_{10,q}+8t_{4,q}=1$

4. $x(p,p+1)-y(p,p+1)+w(p,p+1)-2z(p,p+1)-6T_{3,p}+T_{6,P}-2T_{4,P}=2$

5. $Z(p(p+1),p)+W(p(p+1),p)-8T_{3,p}^2-T_{4,p}^2-6P_p^4-2T_{5,p}+3T_{4,p}=0$

6. $z(2^{2n},1)+w(2^n,1)=j_{2n+2}+j_{2n}$

7. $x(2^n,1)+y(2^n,1)=4KY_n-4j_{2n}$

8. The triple $(x(p,p),y(p,p),z(p,p))$ satisfies the homogeneous cone $Y^2-X^2=2Z^2$

2.2: Pattern II:

In (2), the choice $v=2V,u=2U$ (5)

gives

$$U^2+2V^2=w^2 \qquad (6)$$

After performing a few calculations, the integral solutions of (1) are obtained as

$$\left.\begin{aligned} x&=4p^2-2q^2+4pq\\ y&=4p^2-2q^2-4pq\\ z&=4pq\\ w&=2p^2+q^2 \end{aligned}\right\} \qquad (7)$$

Note: Replacing q by p and p by q, x by $-x$ and y by $-y$, in (7), we obtain the solutions of pattern (1).

The above solution set (7) satisfies the following properties:

1. $x(p,p)+y(p,p)+z(p,p)+w(p,p)=22T_{3,p}+11T_{6,p}-22T_{4,p}$

2. Each of the following expression is a nasty number:

(a). $3[4w(p,q)-x(p,q)-y(p,q)]$.

(b). $x(2^n,1)-y(2^n,1)+z(2^n,1)-6j_{2n+1}$

3. $x(p+2,p+1)-y(p+2,p+1)-16T_{3,p}+32T_{5,P}-48t_{4,P}$ is a biquadratic integer.

4. $20T_{3,p} - x(p,1) - y(p,1) - z(p,1) - w(p,1) \equiv 0 (\bmod 3)$

5. $x(2^n,1) - y(2^n,1) + z(2^n,1) + w(2^n,1) - 42J_{2n} \equiv 0 (\bmod 5)$

6. The triple $(x(p,p), y(p,p), z(p,p))$ satisfies the homogeneous cone $X^2 - 2Z^2 = Y^2$

7. $x(p(p+1),p) - y(p(p+1),p) - z(p(p+1),p) + w(p(p+1),p) = 8P_p^5 + 2T_{4,p^2} + 12P_p^4 - 6T_{3,p} + PR_p - T_{4,p}$

In addition to the above two patterns, there are two more patterns of solutions (1) which we present below.

2.3: Pattern III:

Assume $w = p^2 + 2q^2, p, q \neq 0$ (8)

Write 4 as

$$4 = \frac{(2+4i\sqrt{2})(2-4i\sqrt{2})}{3^2} \quad (9)$$

Using (8) & (9) in (3) and applying the method of factorization define:

$$(u + i\sqrt{2}v) = \frac{(2+4i\sqrt{2})(p+i\sqrt{2}q)^2}{3}$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} u &= \frac{2}{3}(p^2 - 2q^2 - 8pq) \\ v &= \frac{4}{3}(p^2 - 2q^2 + pq) \end{aligned}\right\} \quad (10)$$

In view of (2) and (10) the solutions of (1) are obtained as

$$\left.\begin{aligned} x &= 18(p^2 - 2q^2 - 2pq) \\ y &= -6(p^2 - 2q^2 + 10pq) \\ z &= 12(p^2 - 2q^2 + pq) \\ w &= 9(p^2 + 2q^2) \end{aligned}\right\} \quad (11)$$

2.4: Pattern IV:

Consider (6) as

$$U^2 + 2V^2 = 1 \times w^2 \quad (12)$$

Take 1 as

$$1 = \frac{(7+i4\sqrt{2})(1-i4\sqrt{2})}{9^2} \qquad (13)$$

Using (8) & (13) in (12) and applying the method of factorization define:

$$(U + i\sqrt{2}V) = \frac{(7+i4\sqrt{2})(p+i\sqrt{2}q)^2}{9} \qquad (14)$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned} U &= \frac{1}{9}(7(p^2-2q^2)-16pq) \\ V &= \frac{1}{9}(4(p^2-2q^2)+14pq) \end{aligned}\right\} \qquad (15)$$

In view of (2), (5) and (15) the integral solutions of (1) are found to be

$$\left.\begin{aligned} x &= 2(11p^2-22q^2-2pq) \\ y &= 2(3p^2-6q^2-30pq) \\ z &= 2(4p^2-8q^2+14pq) \\ w &= 9(p^2+2q^2) \end{aligned}\right\} \qquad (16)$$

3. Conclusion:

It is to be noted that, instead of (13), one may write 1 as

$$1 = \frac{(1+i2\sqrt{2})(1-i2\sqrt{2})}{3^2}$$

Following the procedure presented above, the corresponding integral solutions of (1) are obtained.

REFERENCES:

1. L.E.Dickson, History of Theory of Numbers, Vol.11, Chelsea Publishing company, New York (1952).

2 .L.J.Mordell, Diophantine equations, Academic Press, London(1969).

3. Carmichael ,R.D.,The theory of numbers and Diophantine Analysis,Dover Publications, New York (1959)

4 .M.A.Gopalan & A.Vijayashankar, *An Interesting Dio.problem* $x^3 - y^3 = 2z^5$, Advances in Mathematics, Scientific Developments and Engineering Application, Narosa Publishing House, Pp 1-6, 2010.

5 .M.A.Gopalan & A.Vijayashankar, *Integrated solutions of ternary quintic Diophantine equation* $x^2 + (2k+1)y^2 = z^5$,International Journal of Mathematical Sciences 19(1-2), 165-169,(jan-june 2010)

6. M.A.Gopalan & A.Vijayashankar, *Integrated solutions of non-homogeneous quintic equation with five unknowns* $xy - zw = R^5$, Bessel J.Math.,1(1),23-30,2011.

7. M.A.Gopalan & A.Vijayashankar, *solutions of quintic equation with five unknowns* $x^4 - y^4 = 2(z^2 - w^2)P^3$,Accepted for Publication in International Review of Pure and Applied Mathematics.