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"NONSTANDARD MODEL AND ITS AXIOMATIC REPRESENTATION"

BABAN KUMAR
Research Scholar
Department of Mathematics, B.N.M.U Madhepura

Dr. Mukund Kumar Singh
Associate Professor and HOD
Dept. of Mathematics, B.N.M.U Madhepura

Abstract

Our main aim is to extend ZFC by replacing the regularity axiom to overcome the hurdle that every nonstandard theory has to deal with Hrback paradox. A. Robinson and E. Zakon initiated super structure approach to Nonstandard analyses. we show here that the Harback paradox may be dated in the universes of nonstandard set theory unlimited. The axiomatic theory *ZFC is an extension of the traditional framework of set-theoretic strucre.

Keywords: (Nonstandard axiom of choice embedding & Harback paradox, ordinal, transfinite induction)

Introduction

We consider here Zermelo-Fraenkel set theory with choice, but do not assume regularity. An ordinal is defined as a transitive set which well-ordered by *2*. A cardinal is an initial ordinal classes which are used to denote extensions of formulas. A proper class is a class which is not a set. V denote the universal class of all sets; WF the class of wellfounded sets; and ON the class of ordinals. The theory *ZFC is formulated in the first-order language consisting of a symbol $\in$ for the membership relation and of a symbol * for the nonstandard embedding. Its axioms are categorized into group theory. The separation and replacement schemata are also assumed for formulas cointaining the * symbol.

1.2 Theorem

There is a linearly ordered class λ and a rank function $R : V \to \lambda$ that satisfies:

(i) $R(x) = \sup\{R(x') + 1 : x' \in x\}$ if $x \neq \emptyset$ and $R(\emptyset) = 0$.

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(ii) $R(^*a) = {}^*\rho(a)$ for every $a \in S$.

(iii) $R(\xi) = \xi$ for all internal-ordinals $\xi \in {}^*ON = \cup_{\alpha \in ON} {}^*\alpha$.

Proof

We proceed with the familiar von Neumann rank ρ : S → ON defined on the class of wellfounded sets. By application of transfinite induction, we get a rank function for internal sets R : I → *ON where $R(x) = \sup\{R(x') + 1 : x' \in x\}$ if $x \neq \emptyset$ and $R(\emptyset) = 0$. Let R *be* defined as the union $\cup_\alpha {}^*\rho_\alpha$ where $\rho_\alpha : V_\alpha \to \alpha$ is the restriction of ρ to the first α levels in the hierarchy. Now, let the ordinal α be given. Let us take $\langle ¸\lambda'_\alpha <\rangle$ to be the Dedekind order-completion of the linearly ordered set $\langle {}^*\alpha, \in\rangle$. Then every element $x \in ¸\lambda'_\alpha \setminus {}^*\alpha$ with a copy $ON_x = \{\beta\, x : \beta \in ON\}$ of the ordinals, which is identified with x. For each $a \in WF({}^*V_\alpha) := \cup_{\beta \in ON} V_\beta({}^*V_\alpha)$, define $R_\alpha(a) = R(a)$ if $a \in {}^*V_\alpha$ and $R_\alpha(a) = \sup\{R_\alpha(a')+1 : a' \in \alpha\}$ otherwise. From *(iii)*, it is proved by transfinite induction that $a \in WF({}^*V_\alpha) \to I\ )\ R_\alpha(a) = R(a)$.

Thus R_α is compatible with R. By repeating, it we obtain of λ_α and R_α for all ordinals α. Let us assume $¸\lambda_\beta \subseteq \lambda_\alpha$ when $\beta \leq \alpha$. Since $\{R_\alpha : \alpha \in ON\}$ is a family of pairwise compatible functions, by taking the unions $R = U_\alpha R_\alpha$ and $\lambda = U_\alpha ¸\lambda_\alpha$, the universal rank function $R : V \to \lambda$ is obtained where R satisfies *(i)* by definition of the R_α's. Let us now prove *(ii)*, if $a \in V_\alpha$ for some α, then $R(^*a) = R(^*a) = {}^*\rho_\alpha(^*a) = {}^*[\rho_\alpha(a)] = {}^*[\rho(a)]$. Also for *(iii)*, if $\xi \in {}^*\alpha$ for some ordinal α, using transfinite induction the standard property $\forall x \in \alpha\ \rho_\alpha(x) = x$" and get ${}^*\rho_\alpha(\xi)=\xi$, hence $R(\xi) = \xi$.

1.3 Theorem

For every model $M \vDash ZFC$ and for every M-cardinal k, there is a model $N \vDash ZFC_o$ such that $WF^N = M$ *and* $N \vDash$*"k^+-saturation".*

Proof

We obtain a model $(UD)M \vDash ZFCo$ and an 2-isomorphism $(U_D)^M : M \to (WF^{U}{}_D)^M$. By identifying each $m \in M$ with $(\ominus_D)^M(m)$, it is found that $(U_D)^M$ is the model N.

For any model $M \vDash ZFC^-$, let us denote by $Card^M$ the set of its cardinals, and by Def^M the set of all M-definable cardinals. Hence,

$$Card^M = \{k \in M : M \vDash \text{"k is a cardinal"}\}$$

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$\mathrm{Def}^M = \{k \in \mathrm{Card}^M : M \vDash \forall x\, (\varphi^{WF}(x) \leftrightarrow x = k)$ for some formula $\varphi(x)$ with exactly one free variable.

The sets are linearly ordered by $\in^M$, the M-interpretation of the membership relation symbol. We find that if $M = \mathrm{WF}^N$ is the wellfounded part of a model N, then $\mathrm{Card}^M = \mathrm{Card}^N$ and $\mathrm{Def}^M = \mathrm{Def}^N$. M' is an *elementary end extension* of M, in symbols $M \preccurlyeq_e M'$, if $M \preccurlyeq M'$ (i.e. M is an elementary submodel of M') and for all $a \in M$, $M' \vDash b \in a$ implies $b \in M$. Let us assume that M and M' are models of ZFC$^-$. If $M \preccurlyeq M'$ then $\mathrm{Def}^M = \mathrm{Def}^{M'}$; and if $M \preccurlyeq_e M'$ then $\langle \mathrm{Card}^M; \in M\rangle$ is an initial segment of $\langle \mathrm{Card}^{M'}; \in M'\rangle$, as linearly ordered sets. It shows that every model of ZFC can be extended to a model of *ZFC, which gives a strong foundational justification to *ZFC.

1.4 Theorem

Every model M ⊨ ZFC has an elementary end extension $M' = \mathrm{WF}^N$ which is the wellfounded part of some model N ⊨ *ZFC. Moreover, if Def^M is not cofinal in Card^M, then one can take M = M'.

Proof

If Def^M is cofinal in Card^M, Card^M has countable cofinality and so, by a classic result of H.J. Keisler and M. Morley [KM], there exists a proper elementary end-extension $M' \succcurlyeq_e M$. If Def^M is not cofinal in Card^M, let M' = M. In both cases, it is possible to pick $\succcurlyeq\ \in \mathrm{Card}^{M'}$ such that M' ⊨"$v > k$" for all $k \in \mathrm{Def}^M$. Now apply the previous theorem by considering the M'-cardinal v and get a model N ⊨ $*\mathrm{ZFC}_0$+"k^+-saturation", where $\mathrm{WF}^N = M' \succcurlyeq_e M$. Notice that N satisfies the axiom schema of saturation. In fact, let $k \in N$ be a "definable" cardinal, that is $k \in \mathrm{Card}^N$ and $N \vDash \forall x \in S\ (\varphi^S(x) \leftrightarrow x = k)$. Then $k \in \mathrm{Def}^N = \mathrm{Def}^{M'} = \mathrm{Def}^M$, hence N ⊨"$v > k$" and N ⊨"k-saturation".

1.5 Theorem

ZFC is interpretable in *ZFC by relativizing quantifiers to the class of standard sets. In other words for every $\in$-sentence σ

$$ZFC \vdash \sigma \Leftrightarrow,\ *ZFC \vdash \sigma^S$$

Proof

If ZFC $\forall$ σ, then there is a model M ⊨ ZFC + ¬ σ. By an application of assertion of theorem, we get a model N ⊨ *ZFC with $WF^N \succcurlyeq_e$ M. we thus obtain that M ⊨ ¬σ ⇔ N ⊨ $(\neg\, \sigma)^{WF}$. Which prove also that $(\neg\, \sigma)^{WF}$ is the same as : ¬ $(\sigma)^{S}$.

Thus N ⊨ *ZFC + $\neg(\sigma)^S$ and this proves *ZFC ⊢/ σ^S. The reverse implication may be proved in similar way. If *ZFC ⊢/ σ^S, then there is a model N ⊨ *ZFC+:$(\sigma)^S$. In particular, WFN ⊨ ZF^C + ¬ σ, hence ZFC ⊢/ σ. Hence, the theorem is prove.

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