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STUDY OF CHARACTERIZATION OF SPECTRAL MEASURES ON BANACH SPACE AND ITS IMPACT

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Abstract

We derive some properties of co-relation operator and its spectral measure. We derive theorems related to Lebesque decomposition theorem and Hellinger integrals. The stationary sequence play a crucial role in determining spectral measure and angle between the part and future of a stationary sequence. we derive here the characteristic properties of the spectral measures on stochastic processes and its regularity condition. We discuss characteristic properties of the regularized version of a stochastic process equivalent to the modified application of random variables and its application to Ito's construction of simple paths of a given process. The significant aspect in Ito′s construction consists of the regularization of the sample paths of the given process.

Keywords:- co-relation operator, characteristic properties, cylindrical probability, Gaussian measures, separable Banach spaces, Gaussian random element,

Introduction

Let (B, (B)μ) be a real separable Banach space with a man –zero Gaussian measures μ defined on the Borel σ-algebra (B). we know that for these spaces the measure μ may be obtained as a □-extension of the canonical cylindrical probability measure $H_{\square}$ concentrated on a separable Hilbert space, $H_{\square}$ such that

$$B* \subset H_\mu^* = H_\mu \subset B \qquad \ldots(1.1)$$

The embedding of $H_{\square}$ into B is a continuous linear injection, and each continuous linear functional on B restricted to the space $H_{\square}$ □in the norm of B.

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The problem for separable Banach spaces was considered by L. Gross and J. Kuelbs, H. Sato. These results were generalized to locally convex Hausdorff for real vector spaces by Ch. Borell. In spaces L_ϕ, the canonical measureable stochastic process play an important role. The elements of L_ϕ are classes of almost measurable equivalent functions. We simplify this problem in L_ϕ spaces by considering the canonical measurable stochastic process ξ (t). we consider here that for each fixed x. ξ(t) is defined for almost all t. in the space L_ϕ the canonical measurable process play s the same role as the evolutions $\prod_t$ in the space C(T). we modify here the results of A. T. Bawniczak in more simple and elegant. We extend here the results derived by J. Rajput.

(i) Definition (linear space)

Let us denote by (x, F, □) a measurable linear space with complete □-algebra, F in a probability measure □ and by (R, 𝔅) the real line with Borel □-algebra of subsets.

(ii) Definition (Quasi –additive)

F is a quasi-additive measureable functional (q.m.f) defined on (X, F) if

(i) F is a measurable mapping from (X, F) into (R, 𝔅),

(ii) F(x ± y) = F(x) ± F(y) [□ x □] a.e

We state the following significant direction (i) Each finite linear combination of q. m. f. is a q.m.f. Let F be a q.m.f. we identify f with the equivalence class of measurable functions equal to $F_{□}$ a.e., and by $X^*_{□}$ we denote the space of all q.m.f. with topology generated by convergence in measure (ii) The space $X^*_{□}$ is a linear space with the ordinary operations of addition and scalar-multiplication.

(iii) Definition (Gaussian random element)

A random element Y on (X F) is a Gaussian random element if for each arbitrary independent random elements Y_1, Y_2 with the same distribution as Y, the random elements $Y_1 + Y_2$ +, $Y_1 - Y_2$ are independent. A probability measure □ defined on measureable linear space (X, F) is a Gaussian measure if □ is the distribution of a Gaussian element Y. we now establish a lemma related to Gaussian measures. we state a lemma before proving the main theorem.

Lemma

Let (X, F, □) be a measurable linear space with Gaussian measure □; then each quasi-additive measureable functional F has a Gaussian distribution. when □ is a symmetric measure then F has a symmetric distribution.

Theorem

Let (X, F, □) be a measurable linear space with □-algebra F completed in a symmetric Probability measure □. Also $\{F_\tau : \tau \in \Gamma\}$ of quasi-additive measurable functional generates F, then

$$\overline{L\iota n\{F_\tau : \tau \epsilon \Gamma\}} = X_\mu^* \qquad \ldots(1.2)$$

Where the closure of $\text{lin}\{F_\tau : \tau \in \Gamma\}$ is taken with respect to convergence in measure.

Proof

It is sufficient to prove that

$$\text{t } \mathcal{L}(\Gamma)\overline{L\iota n\{F_\tau : \tau \epsilon \Gamma\}} \supset X_\mu^*$$

Let us suppose that there exists a g.m.f. F_0 such that $F_0 \in X^*_{□}$ and $F_0 \notin L(\Gamma)$. Since $L(\Gamma)$ is a closed linear subspace of L_2(□), then there exists an orthogonal projection on this subspace

$$P(G) = Proj\ L_{(\Gamma)}\ (g) \qquad \ldots(1.3)$$

From assertion of previous theorem, it follows that $F_0 - P(F_0)$ is a non-degenerate Gaussian random variable, which is independent of F and measureable with respect to F, this is a contradiction thus the theorem is proved we derive properties of probability measure in Orlicz space. Let (T, F, m) be an arbitrary □-finite measure space with □algebra F. Let S be the space of equivalence classes of all real –valued measurable functions with convergence in measure m. Let S_0 be the subset of S consisting of elements equivalent to

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functions of the form 1_E, where $E \in F$, $m(E) < \infty$. We assume here that S_0 is separable with respect to the topology induced from S. Let $L_0 = L_0(m)$ be the closed linear subspace in S generated by S_0. Let ϕ be continuous, non-negative, non-decreasing function defined for $u \geq 0$ such that $\phi(u) = 0$ if and only if $u = 0$. we also that the function $\phi(u)$ satisfies the condition (1.3), i.e., there is a positive constant k such that

$$\phi(2u) \leq k\phi(u)$$

For $x \in S$, let us put

$$R_{\phi}(x) = \int_T \phi(|x(t)|)\ m\ (dt) \qquad \ldots(1.4)$$

and let L_ϕ be the set of all $x \in S$ such that $R_\phi(ax) < \infty$ for a positive constant a. The set L_ϕ is a linear space under the usual addition and scalar multiplication. Hence, it becomes a complete metric linear space under the semi norm $||\ .\ ||_\phi$:

$$||x||_{\phi} = inf\{c : c > 0,\ R_{\phi}\ (C^{-1}\ x)\ \ \leq c\} \qquad \ldots(1.5)$$

we extend it to Orlicz space to Weiner space that $L_\phi \subseteq L_0$ (endowed with the Borel □-field induced by the topology of convergence in measure m). it is denoted by $\mathfrak{B}\ (L_\phi)$, $\mathfrak{B}\ (L_0)$ the Borel □-algebras in L_ϕ, L_0 respectively, then

$$\mathfrak{B}(L_{\phi}) - L_{\phi} \cap \mathfrak{B}\,(L_0)$$

We illustrate Orlicz spaces by examples

(1) If $\phi(u) = u/1+u$; $T = [0, 1]$ and m is Lebesgue measure on F the □-algebra of all Lebesgue measurable sets, then $L_\phi = L_0[0, 1]$.

(2) If $\phi(u) \in u^p$, $0 < p < \infty$ and $T = [0, 1]$ with m Lebesgue measure on F the □-algebra of all Lebesgue measureable sets, then $L_\phi = L_P\ [0, 1]$.

Let $\xi = \{\xi(t) : t \in T\}$ be a measurable stochastic process defined on a probability space (Ω, Σ, P), such that $\xi(., \omega) \in L_\phi$ a.e. Let $\tilde{\xi} : \Omega + L_\phi$ be defined as follows.

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$$\xi = \begin{cases} \xi(.\,\omega) & if\ \xi(.,\omega) \in L_\phi \\ 0 & if\ \xi(.,\omega) \notin L_\phi \end{cases} \qquad \ldots(1.6)$$

Since $\tilde{x}$ has a separable range, it is a random element (r.e) with values in L_ϕ. The probability distribution of $\tilde{x}$is denoted by □□ and called the measure induced by the process □.

Modified version of Probability measures on stochastic process

We suitably apply the techniques for proving earlier results which depend on properties of locally convex vector spaces. Here the dual space of continuous linear function play an important role. However, there are metric linear spaces, natural from the Point of view of probability theory, having no non-trivial continuous linear functional. A stochastic process □ = {□(t) : t ∈ T} is called Gaussian if there is a measureable subset T_0, $m(T_0) – 0$ such that for every t_1,...., $t_k \in T\backslash T_0$; <□(t_1),... □(t_k)> is a Gaussian random vector. Let □□be a Gaussian mesure on a separable Orlicz space (L_ϕ, (L_ϕ)). Then there exists a measurable Gaussian stochastic process □ on T x Ω, (Ω, Σ, P) = (L_ϕ , $\mathfrak{B}(L_\phi)$, □) including measure □. Let θ(t) = E□(t), K(s, t) = E(□(s)) - θ(s) (□(t) - θ(t)) then θ ∈ L_ϕ and (for the diagonal of K) $K^{1\backslash 2} \in L_\phi$. T. Bycz knows had shown that for each probability measure □ on (L_ϕ, (L_ϕ)) we construct a measurable stochastic process □ = {□(t) : t ∈ T} with sample paths in L_ϕ such that $\tilde{x}(x)$ = x μ a., e; the induced measure □□ equals □ and for every par (s, u) of real numbers.

$$□(t;\ sx \pm uy) = s□(t,\ x) \pm u□(t \in y)\quad m\ x\ □\ x\ □□□□□\text{a.e.} \qquad \ldots(1.7)$$

We thus find that if □ = {□(t) : t ∈ T} is a measurable stochastic process which satisfies assertion of theorem. By including measure □ on a separable Orlicz space (L_ϕ, $\mathfrak{B}(L_\phi)$) and $T_0 \in F$ is an arbitrary subset, $m(T_0) = 0$, then there exists a countable family {□(t_i) : $t_i \in T/T_0$} of quasi additive measureable functional obtained from the process □ which generates the □□algebra $\mathfrak{B}$ (L_ϕ) (mod □) (i.e., there exists N ∈ $\mathfrak{B}(L_\phi)$, □(N) = 0) such that the familiry {□(t_i) : $t_i \in T\backslash T_0$} generates $N^c \cap \mathfrak{B}(L_\phi)$ in $N^c \cap L_\phi$. Let $□_I$ be some measurable stochastic process with sample paths in L_ϕ ; i=1, 2 and let $□_I$ denote the measure induced by $□_I$ on L_ϕ, i =1, 2. Then $□_1 = □_2$ iff there is a T_0 ∈ F. m (T_0) = 0 such that the corresponding finite dimensional distributions of $□_\phi$ and $□_\vee$ based on points $T\backslash t_0$ are equal. Let □ = {□(t) : t ∈ T} be an arbitrary stochastic process of the second

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order. By $H_s(\square)$ we will denote the subspace of $L_2(\Omega, \Sigma, P)$ generated by the set of random variables $\{\square(t) : t \in S\}$. we now establish a related to non-degenerate Gaussian measure.

Theorem

If $\square$ is a non-degenerate Gaussian measure on $(L_\phi, (L_\phi))$, then the space $(L_\phi)^*{}_\square$ is a non-trivial closed liner subspace of $L_2(L_\phi), \mathfrak{B}(L_\phi), \square)$. Let $\square = \{\square(t) : t \in T\}$ be an arbitrary measurable stochastic process defined on $(\Omega, \Sigma, P) = (L_\phi, \mathfrak{B}(L_\phi), \square)$, inducing the measure $\square$, such that $\tilde{x}(x) = x\ \square$ a.e.

$$\square(t, x \pm y) = \square(t, x) \pm \square(t, y) \quad m\ x\ \square\ x\ \square\square\square\square\square\square\square\square\square\square\square\square$$

Then there exists a measurable subset T_0 $m(T_0) = 0$ such that

$$H_T\backslash T_0(\square) = (L_\phi)^*{}_\square \qquad \ldots(1.8)$$

Proof

Let $\square\square\square\{\square(t) : t \in T\}$ be a stochastic process as asserted in the theorem. Then there exists a measurable subset $T_0, m(T_0) = 0$. And a measurable subset N. $\square(N) = 0$, such that $\{\square(t) : t \in T\backslash T_0\}$ is a family of q. m. f defined on L_ϕ which separates points of $L_\phi \cap N^c$. since the measure $\square$ is non-degenerate then the space $(L_\phi)^*{}_\square$ $\square$is non-trivial thus, it follows that

$$H_T \backslash T_0\,(\xi) = \overline{\imath n\,\{\xi(t){:}\ t\ \in T\backslash T_0\}} = (L_\phi)^*_\mu \qquad \ldots(1.9)$$

The comparative analysis of Gaussian measure vrs Probability measure

Let $\square$ be a Gaussian measure on a separable Banach space (E (E)), if and only if each continuous linear functional on E is a Gaussian real random variable, for a Banach space it is equivalent to the definition of quasi-additive linear functions. We replace linear functional by the adjoint space of a vector space by a space where as defined as follows.

Theorem

Let (L_ϕ, (L_ϕ), □) be a separable Orlicz space with a probability measure □ and the dual space $(L_\phi)^*$□. Then □□is a Gaussian measure if and only if every q.m.f. F is a Gaussian real valued random variable.

Proof

If □ is a Gaussian measures on (L_ϕ, (L_ϕ)), then for every q.m.f. F is a Gaussian real valued random variable. Let □ be an arbitrary measure on (L_ϕ, (L_ϕ)) and let us assume that each q.m.f. F is distributed. Hence, we thus find that there exists a measurable stochastic process. □ = {□(t : t ∈ T)} defined on (L_ϕ, $\mathfrak{B}(L_\phi)$, □) such that $\tilde{x}(x)$ = x □ a.e., and there exists a measurable subset T_0, $m(T_0) = 0$ such that {□(t) : t ∈ $T \backslash T_0$} is a family of q.m.f. The assumption that any finite dimensional vector <□(t_1), ... □(t_n)>, {t_1,....,t_n} ⊂ $T \backslash T_0$ is Gaussian distributed It implies that the measure □ is Gaussian. We further obtain that for any probability measure □□defined on (L_ϕ, $\mathfrak{B}(L_\phi)$,) there corresponds a measurable stochastic process □□□□□□(t): t ∈ T} with almost all sample paths in L_ϕ, inducing measure □, i.e. $\tilde{x}(x)$ = x □ a.e. let us consider the case where T is the real line R, F = B(R) and m is Lebesgue measure; for a class of measurable stochastic process with sample paths in L_ϕ including measure □, there is one "nice" stochastic process, □ = {□(t) : t ∈ T}, called a canonical measurable stochastic process with is of great impotence. The advantage of using this process lies in the fact that for each equivalence class x ∈ L_ϕ, □(x) ∈ x(t). if x(t) is a continuous function then □(x) = x. it developed the canonical measurable stochastic process □ = {□(t) : t ∈ T} based on the Lebesgue density theorem which stats that each measurable, real – valued function defined on the real line T with Lebesgue measure m is almost everywhere continuous in the density topology d. By taking the approximate limit of elements of L_0. L to obtained as selector of L_0 of "nice functions" on T, the canonical measureable function on T. Hence, for a given measurable stochastic process □ = {□(t) : t ∈ T} the canonical modification is obtained by means of the regularization of the sample paths. ILo also proved that when a stochastic process is defined on standard Borel space the canonical modification of it is a measureable stochastic process. We extend its area of application by establishing a lemma.

$$\xi(t,w) = \begin{cases} \lim\limits_{\substack{s\to t \\ d}} \xi(s,w) \, if \ this \ limit \ exists \\ undefined \quad if \ \ otherwise \end{cases} \qquad \ldots(1.10)$$

Let us assume that (Ω, ∑, □) = (L□, B(L□), μ). Let ξ be function defined on (L□, B(L□), μ) as follows .

ξ(x) = x^a, i.e.

$$x^a(t) = \begin{cases} \lim_{\substack{s \to t \\ d}} x(s) \, if \; this \; limit \; exists \; and \; is \; finite \\ undefined \quad if \;\; otherwise \end{cases} \qquad \ldots(1.11)$$

The value of ξ(x) at time t ∈ T, if it is defined and denoted by ξ(t,ω). Since (L□, B (L□)μ) is a standard Borel space, then by Ito's theorem {ξ(t) : t ∈ T} is a canonical measurable stochastic process such that for each equivalence class x ∈ L□, ξ(., x) ∈ x, the distribution μ_ξ of the random element $\bar{\xi}$ equals μ, and Fubini's Theorem implies that for almost all t ξ(t, x ± y) = ξ(t, x) ± ξ(t, y) μ x μ a.e. if the probability measure μ is Gaussian then the stochastic process {ξ(t) : t ∈ T} is a Gaussian process. We have used here the symbol ξ for the canonical measureable random function defined on (L□ B□L□)μ) by ξ(x) = x^a, and we also denote by {ξ(t) L t ∈ T} the canonical stochastic process obtained from it.

Interpretation on spectral measure

Let X be an L(H, K)- valued stationary sequence and let E(.) be the spectral measure of the shift operator of X. The $L^+(H)$ – valued function defined as follows.

$$F(\square) = X(0)^* \, E(\square) \, X(0) \qquad \ldots(1.12)$$

is said to be the spectral measure of the sequence X. According to derivation of H. Shelah. F(.)x is a countable additive H-valued set function. Thus for every bounded Borel measureable function *f* the integral *f(t)F(dt)x* is well defined (3.12) implies that.

$$\Gamma(n)x = \int e^{int} F(dt)x \qquad \ldots(1.13)$$

For every x □ H and n □ Z. we modify the result due to on spectral measure of stationary sequence.

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