International Journal of Engineering, Science and Mathematics
Vol. 15 Issue 8, Aug 2026,
ISSN: 2320-0294 Impact Factor: 8.215
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

STUDY OF SOME POLYNOMIAL IDENTITIES BUYING COMMUTATIVITY FOR RINGS

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Abstract

We know that a ring R is commutative if and only if [x, y] =0 for all x, y ∈ R. It is natural to question whether a ring in which the commutators [xy, yx] are identically zero, be necessarily commutative. In 1970, Gupta [38] proved that a division ring is commutative if and only if [xy, yx] = 0. Earlier, Israel N. Herstein [45] established that a division ring D in which xy-yx is central for every pair of elements x,y ∈ D, must be commutative. Motivated by these results, Awtar [18] and Quadri [33] proved that a semi prime ring with either of [xy, yx] or xy o yx central, must also be commutative.

$$R = \left\{ aI + D \,\middle|\, D = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ a,b,c,d \in GF(2) \right\}$$

The ring rules out the possibility of extending the mentioned results for arbitrary rings. However, in section we extend the results for rings with unity 1 by imposing certain torsion conditions on the commutators.

Keywords:- Non-commutative ring, semi prime ring, commutators, arbitrary rings, upper triangular matrices, polynomial identities, commutativity,

Introduction

In [61], M.A. Khan et. al. considered certain fairly general assumptions which rendered a semi prime ring commutative. The theorem to Which we refer is, namely: A semi prime ring R in Which either of the conditions

$[x^m y^n - xy^n x, x] = 0$ or $[x^s y^t - yx^s y, x] = 0$

holds for all x,y ∈ R, where m,n,s and t are positive integers, turns out to be commutative. In section 3, we carry out this type of investigation further and extend certain previously obtained results.

Long back, Gupta [38] proved that a division ring R is commutative if and only if [xy, yx] = 0 for all x,y in R. Later, Awtar [18] and Quadri [83] generalized the mentioned result

and finally settled that a semi prime ring R with either $[xy, yx]=xy^2 x-yx^2 y$ o~ xy o $yx=xy^2x+yx^2y$ central, is necessarily commutative {cf.[83] , Theorem 2.2).

The example of the non-commutative ring of 3x3 strictly upper triangular matrices over the ring Z of integers rules out the possibility of extending the result for arbitrary rings. Though the mentioned ring does not contain unity, the following example shows that the above theorem is not valid for an arbitrary ring even if it contains unity.

EXAMPLE 1.1

Let $R=\left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a,b,c,d \in Z\right\}$. Then R is a non-commutative ring with unity Which satisfies $[xy, yx] \in Z(R)$. Also, if we replace Z by GF(2) in above example, then R is a non-commutative ring with unity satisfying both the conditions $xy^2 x - yx^2 y \in Z(R)$ and $xy^2 x + yx^2 y \in Z(R)$.

However, by imposing an appropriate torsion restriction on additive group (R, +), we prove the following:

THEOREM 1.1. Let R be a ring with unity 1 in which for all $x, y \in R$, xy o $yx \in Z(R)$. Further, suppose that commutators of R are 2-torsion free. Then R is commutative.

PROOF. Since xy o $yx \in Z(R)$, $[xy$ o $yx, x] = 0$ for all $x, y \in R$ i.e. $[xy^2 x+yx^2 y, x] = 0$, and thus we have (1) $x [y^2, x]x + [yx^2 y, x] = 0$.

On replacing x by $(1 + x)$ in (1) we get,

$[y^2, x]+x[y^2 ,x]+[y^2 ,x] x + x[y^2 ,x]x+[y^2 ,x]+2[yxy, x]+[yx^2y, x]=0$.

This, on using (1), becomes

(2) $x[y^2 ,x]+[y^2 ,x]x+2[y^2 ,x]+2[yxy,x]=0$.

Again on replacing x by $(1 + x)$ in (2) and using (2) 2 we obtain $4[y^2, x] = 0$. Now, in view of the torsion condition on the commutators, we obtain

(3) $[y^2, x] = 0$.

Next, replace y by $(1 + y)$ in (3) and use (3) to get $2[y, x] = 0$. Hence $[y, x] = 0$ and R is commutative.

We prefer to prove it here completely because of its direct relevance with the above discussion and more because of simple computational techniques used.

THEOREM 1.2. Let $n>2$ be a fixed positive integer and R be a ring with unity 1 in which for each x, y in R either $xy^n x - yx^n y$ or $xy^n x + yx^n y$ is central. If commutators in R are n!-torsion free, then R is commutative.

PROOF:- We prove the theorem for the rings satisfying $xy^n x - yx^n y \in Z(R)$ and for the other case the proof will follow similarly.

By hypothesis we get $[xy^n x - yx^n y, x] = 0$ for all x, y in R, and so we have

(4) $x[y^n , x]x = [yx^n y, x]$.

If we replace x by $(1 + x)$ in (4), then we get

$$[y^n , x] + x[y^n , x] + [y^n , x]x + x[y^n, x]x = [y (1 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots\ldots.+ {}^nC_nx^n) y, x],$$

Which, in view of (4), becomes

(5) $$[y^n, x] + x [y^n, x] + [y^n, x] x = [y (l+ {}^nC_1x + {}^nC_2x^2 + \ldots\ldots\ldots.+ {}^nC_{n-1}x^{n-1}) y, x]$$

Again, on replacing x by $(1 + x)$ in (5) and using (5) we find

(6) $$2[y^n, x] = [y \{n + {}^nC_2 (l+ {}^2C_1x) + {}^nC_3 (l+ 3C_1 x + 3C_2 x^2) + {}^nC_4 (l+ {}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3) + \ldots\ldots\ldots.. + {}^nC_{n-1} (l+ {}^{n-1}C_1 x + {}^{n-1}C_2 x^2 + \ldots\ldots\ldots..$$

$+ {}^{n-1}C_{n-2}\, x^{n-2})\}y, x].$

Repeating the same process third and fourth time, we get respectively,

$0 = [y\{ {}^{n}C_2\ {}^{2}C_1 + {}^{n}C_3 (3C_1 + 3C_2 (1 + {}^{2}C_1x)) + {}^{n}C_4 ({}^{4}C_1 + {}^{4}C_2\ (1 + {}^{2}C_1x)$
$+ {}^{4}C_3\ (1 + {}^{3}C_1x + {}^{3}C_2 x^2)) + \ldots\ldots\ldots\ldots\ldots + {}^{n}C_{n-1}\ ({}^{n-1}C_1 x +$
$ {}^{n-1}C_2\ (1 + {}^{2}C_1x)\ {}^{n-1}C_3 (l + {}^{3}C_1 x + {}^{3}C_2 x^2) + \ldots\ldots\ldots\ldots +$
$ {}^{n-1}C_{n-2}\ (1 + {}^{n-2}C_1x + {}^{n-2}C_2x^2 + \ldots\ldots\ldots\ldots + {}^{n-2}C_{n-3}\, x^{n-3}))\}\ y, x]$

and

$0 = [y\ \{ {}^{n}C_3\ {}^{3}C_2 + {}^{2}C_1 + {}^{n}C_4 ({}^{4}C_2\ {}^{2}C_1 + {}^{4}C_3 ({}^{3}C_1 + 3C_2 (1 + {}^{2}C_1x))) + \ldots\ldots$
$+ {}^{n}C_{n-1}\ ({}^{n-1}C_2\ {}^{2}C_1 + {}^{n-1}C_3 ({}^{3}C_1 + {}^{3}C_2\ (1 + {}^{2}C_1x)) + \ldots\ldots\ldots$
$+ {}^{n-1}C_{n-2}\ ({}^{n-2}C_1 + {}^{n-2}C_2x (1 + {}^{2}C_1x) + {}^{n-2}C_3 (1 + {}^{3}C_1 x\ + {}^{3}C_2 x^2) + \ldots\ldots\ldots$
$+ {}^{n-2}C_{n-3} (1 + {}^{n-3}C_1 x + {}^{n-3}C_2 x^2 + \ldots\ldots.. + {}^{n-3}C_{n-4}\, x^{n-4}))\}y, x].$

Hence it is clear that repeating n - times the same process of replacing x by (1 + x) and using the previously obtained identity each time, we finally obtain,

$0 = [y\ ({}^{n}C_{n-1}\ {}^{n-1}C_{n-2}\ {}^{n-2}C_{n-3} \ldots\ldots\ldots\ {}^{2}C_1)\ y, x].$

This gives that $n!\ [y^2, x] = 0$ and in view of the torsion condition on the commutators, we get $[y^2, x] = 0$, which is same as (3) and hence the commutativity of R follows as n>2.

The above result can also be easily established for n = 0 and n = 1 provided the commutators are 2-torsion free.

The existence of enough rings with $R^3 \subseteq Z(R)$ rules out the possibility of extending the above theorem for arbitrary rings. However, it may be extended to semi prime rings as follows:

THEOREM 1.3. Let R be a semi prime ring. If there exists a fixed non negative integer n such that either $[xy^n x - yx^n y, x] = 0$ or $[xy^n x + yx^n y, x] = 0$ for all x, y in R, then R must be commutative.

PROOF:- Let R satisfy either of the conditions $[xy^n x - yx^n y, x] = 0$ or $[xy^n x + yx^n y, x] = 0$. In both cases, R satisfies a polynomial identity with relatively prime integral coefficients. Now the consideration of $x=e_{11}$ and $y=e_{11}$ shows that, for every prime p, the ring of 2x2 matrices over GF (p) fails to satisfy the given identities. Thus in view of Lemma 1.2, of the previous paper, R is commutative.

3. Recently Khan, Quadri and Ali [61] proved that a semi prime ring R satisfying any of the identities $[x^m y^n - xy^n x, x] = 0$ and $[x^s yt - yx^s y, x] = 0$ for all $x, y \in R$, where m, n, s and t are positive integers, turns out to be commutative. Later, Abujabal and Khan [4] proved a similar result for associative rings with unity by imposing a torsion condition on the commutators.

Motivated by the above study, we consider the following ring properties:

(P_1) There exist non negative integers n > 0, k, m, p and r with $n \neq k+p$ for m = r, such that $[x^n y^m - x^k y^r x^p, y] = 0$ for all x, y in R.

(P_1^*) There exist non negative integers k, m, n, p and r with k = 0 or p = 0 for $r \neq 0$, otherwise $m \neq 0$ such that $[x^n y^m - x^k y^r x^p, x]=0$ for all x, y in R.

(P_2) There exist non negative integers k, m, n and p with at least one of k, n and p non zero and $n \neq k+p$ for m = 2 such that $[xy^n x - y^k x^m y^p, x] = 0$ for all x, y in R.

(P_2^*) There exist non negative integers k, m, n and p with at least one of k and p zero for $m \geq 2$ such that $[xy^n x - y^k x^m y^p, y] = 0$ for all x, y in R.

Q (m) For all $x, y \in R$, m[x, y] = 0 implies that [x, y] = 0.

Obviously every ring R has the property Q (l) and every m - torsion free ring has the property Q (m).

We first state the following lemma due to Tong [99], which is used as the main tool for proving our results in this section.

LEMMA 1.1. Let R be a ring with unity 1. Let $I_0^r(x) = x^r$. If k ≥ l, let $I_k^r(x)=I_{k-1}^r(1+x)$ - $I_{k-1}^r(x)$. Then $I_{r-1}^r(x) = \frac{1}{2}$ (r-1) r!+r!x; $I_r^r(x)$= r! and $I_j^r(x)$=0 for j>r.

THEOREM 1.4. Let R be a ring with unity 1, satisfying the property (P_1). Suppose further that R satisfies the property Q (s!), where s=max. (m, n, r, (k + p)).

Then R is commutative.

PROOF:- We use the symbols of Lemma 3.1 to write the property (P_1) as

(7) $[x^n, y]\, r_0^m(y) = [x^k\, r_0^r\,(y)\, x^p, y]$

On replacing y by (1+y) in (7), we get

$[x^n, y]\, 1_0^m\,(1+y) = [x^k\, I_0^r\,(1+y)\, x^p, y]$, which, on using Lemma 3.1 alongwith (7), becomes

$[x^n, y]\, I_1^m\,(y) = [x^k\, I_1^r\,(y)\, x^p, y]$.

Case (i). Let m>r. Then we repeat the process of replacing y by (1+y) in (7) m-times and use the above technique to get $[x^n, y]\, I_m^m\,(y) = [x^k\, I_m^r\,(y)\, x^p, y]$, and so by Lemma 1.1, we obtain m! $[x^n, y] = 0$. Since m! is a divisor of s! and R has the property Q (s!), it is obvious that R also has the property Q(m!) and so $[x^n, y] = 0$. Again, in the symbols of Lemma 1.1, it can be written as

(8) $[I_0^n(x), y] = 0$.

Next, we replace x by (1+x) in (8) and use Lemma 3.1 to find $[I_0^n\,(x) + I_1^n\,(x), y] = 0$, which, on using (8), becomes

(9) $[I_1^n\,(x), y] = 0$.

Now, on replacing x by (1+x) in (9) and iterating (n-2)- times, we get $[I_{n-1}^n\,(x), y] = 0$, which, in the light of Lemma 1.1, yields $[{}^1\!/_2\,(n-1)n! + n!\,x, y] = 0$, that is, n! [x, y] = 0. Thus, by using the torsion condition on commutators, we get [x, y] =0 forcing that R is commutative.

Case (ii). Let m<r. If k = p = 0, then (7) becomes $[x^n, y]\, I_0^m\,(y) = 0$, which, on replacing y by (1+y) and iterating m-times yields $m![x^n, y] = 0$ in view of Lemma 1.1. Thus, by the torsion condition we get $[x^n, y]=0$, that is, $[I_0^n\,(x), y] = 0$, which is same as (8) and hence we obtain the commutativity of R. So, we can assume that either k>0 or p>0. Now the process of replacing y by (1+y) in (7) and iterating r-times, yields

$[x^n, y]\, I_r^m\,(y) = [x^k\, I_r^r\,(y)\, x^p, y]$. Since m<r, we find

$0 = r!\,[x^{k+p}, y]$, i.e. $[x^{k+p}, y] = 0$, which, using the symbols of Lemma 1.1, can be written as $[I_0^{k+p}\,(x), y] = 0$.

Thus we get an identity similar to (8) and so again the commutativity of R can be established.

Case (iii) Let m=r. Then on replacing y by (1+y) in (7) and iterating m-times and using Lemma 1.1, we obtain $m!\,[x^n, y] - m!\,[x^{k+p}, y]$, and by Q(s!) we get

$[x^n, y] = [x^{k+p}, y]$. This, again using the symbols of Lemma 1.1, can be expressed as

(10) $[I_0^n\,(x), y] = [I_0^{k+p}\,(x), y]$.

If n<k+p, then on replacing x by (1+x) in (10), iterating (k+p-1)-times and using Lemma 1.1, we get

(k+p)! [x,y] = 0, which implies that [x, y] = 0 and hence R is commutative. We replace x by (1+x) in (10) and iterate (n-l)-times, in case n>k+p, to get n! [x, y] = 0, which implies the commutativity of R.

Now we prove the following result which, in fact, is a wide generalization of our Theorem 1.2.

THEOREM 1.5. Let R be a ring with unity 1 satisfying (P_2). Further, if R satisfies the property Q(2(s!)), where s= max.(m, n, (k+p)), then R is commutative.

PROOF:- The property (P_2), in the symbols of Lemma 3.1, can be put in the form

(11) $x\,[y^n, x]x = [y^k\, I_0^m\,(x)\, y^p, x]$.

If n=0, then k+p ≠ 0 and so (11) becomes $[y^k I_0^m (x) y^p, x]=0$, which, on replacing x by (1+x) and using Lemma 1.1, becomes

(12) $[y^k I_1^m (x)y^p, x] = 0.$

Now, we replace x by (1+x) in (12), use Lemma 1.1 and iterate (m-l)-times to get m! $[y^{k+p}, x] = 0$, which, by the property Q (2(s!)), implies that $[y^{k+p}, x] = 0$. This, using the symbols of Lemma 1.1, can be written as

(13) $[I_0^{k+p} (y), x] = 0.$

On replacing y by (1+y) in (13), using Lemma 1.1 and iterating (k+p-1)-times we get (k+p)! [y, x] = 0, that is, [y, x] = 0 and hence R is commutative.

So let n>0, then for k=p=0, (11) becomes

(14) $x[y^n, x]x = 0.$

On replacing x by (1+x) in (14) and iterating twice, we get

(15) $2[y^n, x] = 0.$

Thus, by property Q (2(s!)), (15) gives [y, x] = 0, which can be written as $[I_0^n (y), x] = 0$. So we get an identity similar to (13) and the commutativity of R follows.

Henceforth, we may assume that either k>0 or p>0, that is, k+p>0. Now, let m < 2. On replacing x by (1+x) in (11) and iterating twice , we get $2[y^n, x]=[y^k I_2^m (x) y^p, x]$, that is, $2[y^n, x] = 0$, as m<2, which is same as (15) and hence R is commutative.

Now, let us consider the case when m>2. We replace x by (1+x) in (11) and iterate m-times to get $[y^k I_m^m (x)y^p, x] = 0$, that is, m! $[y^{k+p}, x] = 0$, which gives that $[y^{k+p}, x] = 0$ and hence $[I_0^{k+p} (y), x] = 0$, which reduces to (13) and so R is commutative.

Finally, we take the case when m=2 and hence n ≠ k+p. On replacing x by (1+x) in (11) and iterating twice, we obtain

(16) $2[y^n, x] = [y^k I_2^m (x) y^p, x].$

So, $2[y^n, x] = 2[y^{k+p}, x]$, which, using the property Q(2(s!)), becomes $[y^n, x] = [y^{k+p}, x]$. Again, in the symbols of Lemma 1.1, it can be written as

(17) $[I_0^n (y), x] = [I_0^{k+p} (y), x].$

Now if n<k+p, then we replace y by (1+y) in (17) and iterate (k+p-1)-times to get (k+p)! [y, x] = 0, which by Q(2(s!)) becomes [y, x] = 0 and hence R is commutative.

In case n>k+p, we repeat the process of replacing y by (1+y) in (17) and iterating (n-1)-times to get n! [y,x] = 0. Hence, [y,x]=0 and R is commutative. Thus Theorem 1.5 is completely proved.

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