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ON COMMUTATIVITY OF PRIMITIVE RINGS WITH SOME IDENTITIES

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Abstract
We prove some commutativity theorems of non-associative Primitive rings with some identities in the center, We present some preliminary results that we need in the subsequent discussion and prove some commutativity theorems of non-associative rings. We prove that a non- associative primitive ring with $(xy)^2 - xy \in Z\,(R)$ or $(xy)^2 - yx \in Z\,(R)$ for all x, y in R is commutative we prove that if R is a non-associative primitive ring with identity $(xy)^2 - y(x^2y) \in Z\,(R)$ for all x, y in R is commutative. Also we prove that if R is an alternative prime ring with identity $(xy^2)x - (yx^2)y \in$ Z(R) for all x, y in R, then R is commutative.

Keywords:- Primitive rings, alternative prime ring, non-associative rings, Boolean ring, nilpotent, nilring, semiprime ring,

Introduction

1.1 COMMUTATIVITY THEOREMS FOR NON-ASSOCIATIVEPRIMITIVE RINGS WITH $(xy)^2$- $xy \in$ Z(R)

Some commutativity theorems for certain non-associative rings, which are generalizations for the results of Johnsen and others and R.N, Gupta, are proved in this section. Johensen, Outcalt and Yaqub [45] provedthat if a non-associative ring R satisfy the identity $(xy)^2 - x^2y^2$ for all x, y in R, then R is commutative. The generalization of this result proved by R.D.Giri and others [29] states that if R is a non-associative primitive ring satisfies the identity $(xy)^2 - x^2y^2 \in Z(R)$, where Z(R) denoted the center, then R is commutative, A modification of Johnsen's identity viz., $(xy)^2 - x^2y^2$ for all x, y in R for a non-associative ring R which has no element of additive order 2, is commutative was proved by R.N. Gupta [31]. R.D. Giri and others [29] generalized Gupta's result by taking $(xy)^2 - x^2y^2 \in Z(R)$

Now we present the generalizations of these results in the foilwing theorems.

Theorem 1.1:- If R is a non-associative primitive ring with unity satisfying $(xy)^2 - x^2y^2 \in Z(R)$ for all x, y in R, then R is commutative.

Proof:- By hypothesis, $(xy)^2 - x^2y^2 \in Z(R)$.......... .1.1.

Replacing y by $y+1$ in 1.1 and using it, we get

$x(xy) + (xy)x - 2x^2y \in Z(R)$ .... .1.2.

Now replacing x by $x+1$ in 1.2.

We $obtain$ $xy - yx \in Z(R)$.

Therefore $(xy - yx)$ is a two-sided ideal of R. We know that if R is a primitive ring, then R has a maximal right ideal which contains no non-zero ideal of R. Consequently, we obtain $(xy - yx)R$ = (0), which further yields $xy - yx = 0$ due to primitivity of R. This completes the proof of the theorem. 1.1

Theorem 1.2 If R is a 2-torsion free non-associative ring with unity satisfying $(xy)^2 = (yx)^2$then R is commutative.

Proof:- Let x, y ∈ R.

Then $[x(1 + y)^2] = [(1 + y)x]^2$.

i.e., $(x + xy)^2 = (x + yx)^2$

i.e., $x^2 + x(xy) + (xy)x + (xy)^2 = x^2 + x(yx) + x + (yx)^2$

i.e, $x(xy) + (xy)x = x(yx) + (yx)x.$ .... 1.3.

Replacing x by $1 + x$ in 1.3, then we get

$$(1 + x)\ (y + xy) + (y + xy)(l + x) = (l + x)\ (y + yx) + (y + yx)(l + x).$$

Therefore, $y + xy + xy + x(xy) + y + yx + xy + (xy)x = y + yx + xy + x(yx) + y + yx + yx + yx + (yx)x.$

Using 1.3, we get $2(xy - yx) = 0, i.e., xy = yx$. Hence R is commutative.

Theorem 1.3 If R is a 2-torsion free non-associative primitive ring with unity such that $(xy)^2$-$(xy)^2$ ∈ Z(R), for all x, y in R, then R is commutative.

Proof:- Given $(xy)^2$-$(xy)^2$ ∈ Z(R) .... 1.4.

Replacing y by $y + 1$ in 1.4 and using 1.4, we obtain

$x(xy) + (xy)x - x(yx) - (yx)x \in Z(R).$ .... 1.5.

Now replacing x by $x + 1$ in 1.5 and using 1.5, we achieve, $2xy - 2yx \in Z(R)$.

i.e., $2(xy - yx) \in Z(R)$.

Since R is a 2-torsion free ring, $xy - yx \in Z(R)$, Now using the same argument as in the last paragraph of the proof of the theorem 1.1, we conclude that R is commutative.

Now we present, some examples to see that the unity and 2-torsion free are essential in theorems 1.1, 1,2, and.1,3.

Example - 1.1 The conditions (($(xy)^2 - x^2y^2 \in Z(R)$ and $(xy)^2 - (yx)^2 \in Z(R)$ of theorems 1.1 and 1.2 along with the common of primitivity of R are indispensable.

This can be checked in the ring R = $\left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \Big/ a, b, c, d \in Z \right\}$

Example-1.2 The conditions for the ring R, having unity in the theorems 1.1.and 1.2 cannot be dropped. This can easily be verified for the ring

R = $\left\{ \begin{bmatrix} a & a & a \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix} \Big/ a, b \in Z \right\}$

Example- 1.3 The restriction on R, being 2-torsion free in theorem 1.2 is essential one. For if we consider the ring R of quaternions over the field of order 4 namely splitting field of $x^2 + x + l$ over Z_2, then it is not of 2-torsion free but satisfies the identity of theorem 1,2. Yet it is non-commutative.

Example-1.4 Theorem 1.3 is false for rings without unity. In fact any nilpotent ring of index <4 and any nilring of index 2 will trivially satisfy $(xy)^2$-$(xy)^2$, but such rings maynot be commutative. As an example let F be any field. Define an algebra A over F with basis $\{a, b, c\}$, where $ab = c$, all other products zero. A is nilpotent ofindex 3, A is not commutative.

2. COMMUTATIVITY THEOREMS FOR NON-ASSOCIATIVE PRIMITIVE RINGS WITH $(xy)^2 - xy \in Z(R)$

It is well known that a Boolean ring satisfies $x^2 = x$, for all $x \in R$ and this implies commutativity. Similarly we can see the properties of rings in which $(xy)^2 = xy$ for each pair of elements $x, y \in R$. In [54] Quadri and others proved that an associative semiprime ring in which $(xy)^2 - xy \in Z(R)$ is commutative. In this direction we prove that a 2-torsion free non-associative ring with unity satisfying $(xy)^2 - xy \in Z(R)$ or $(xy)^2 - y\,x \in Z(R)$ is commutative. We give an example to show that the unity is essential in the hypothesis. Also, we prove that a non-associative primitive ring (not necessarily having unity) satisfying $(xy)^2 - xy$(or) $(xy)^2 - y\,x$ is central for all x, y $\in$ R is commutative.

First we prove the following theorem:

Theorem 2.1 Let R be a 4-torsion free non-associative ring with unity satisfying (i) $[(xy)^2 - xy, y] = 0$ or (ii) $[(xy)^2 - xy, x] = 0$. Then R is commutative.

Proof:- Hypothesis (i) yields $[(xy)^2 - xy\,]$y = y $((xy)^2 - xy\,]$. .... 2.1.

Putting $x + 1\ for\ x$ in 2.1

$[(xy)y]y + [y(xy)]y = y[y(xy)] + y[(xy)y]$. .... 2.2.

Now replacing $y\ by\ y + 1$ in 2.2 and simplifying it by keeping the brackets unchanged, we obtain

$[(xy)y] + [y(xy)]y + 4(xy)y + y(xy) + (yx)y + 5xy + yx + 2x = y[y(xy)] + y[(xy)y] + (xy)y + 4y(xy) + 3xy + 3yx + y(yx) + 2x$.

……………………………………………………….... 2.3.

Now using 2.2 in equation 2.3, we get

$3(xy)y + (yx)y + 2xy = 3y(xy) + y(yx) + 2yx$. .... 2.4.

Expanding 2.4 by replacing $y\ by\ y + 1$, we get

$7xy + yx = 5yx + 3xy$.

i.e., $4(xy - yx) = 0$.

But R is a 4-torsion free. Hence xy - yx =0, i.e., xy = yx. Thus R is commutative.

Similarly, R is commutative if R satisfies (ii).

Theorem 2.2 Let R be a 2-torsion free non-associative ring with unity satisfying $(xy)^2 - xy \in Z(R)$ for all x, y in R. Then R is commutative.

Proof:- By hypothesis $(xy)^2 - xy \in Z(R)$ .... 2.5.

Replacing $x\ by\ x + 1$ in 2.5,

$$(xy)y + y(xy) + y^2 - y \in Z(R)$$

Again replacing $x\ by\ x + 1$ in 2.6 and using it, we obtain

$2y^2 \in Z(R)$ ……………………… 2.6.

Since R is a 2-torsion free,

$y^2 \in Z(R)$. .... 2.7.

Replacing $y\ by\ xy$ in 2.7, we get

$(xy)^2 \in Z(R)$ . ... 2.8.

But by hypothesis $(xy)^2 - xy \in Z(R)$, hence we get

$xy \in Z(R)$ .... 2.9.

Now again replacing $x\ by\ x + 1$ in 2.9, we get

$xy + y \in Z(R)$ .... 2.10.

From the equations 2.9 and 2.10 we obtain $y \in Z(R)$ for all $y \in R$.

Hence R is commutative.

Theorem 2.3 Let R be a 2-torsion free non-associative ring with unity satisfying $(xy)^2 - yx \in Z(R)$ for all x, y in R. Then R is commutative.
Proof:- Given $(xy)^2 - xy \in Z(R)$ .... 2.1 1.
Replacing x by $x+1$ in 2.11, we get
$(xy)y + y(xy) + (y)^2 - y \in Z(R)$ . ...2.12.
Again replacing x by $x+1$ in 2.12
We obtain $2(y)^2 \in Z(R)$
Since R is a 2-torsion free, then $(y)^2 \in Z(R)$ .... 2.13.
Now replacing y by xy in 2.13, we get
$(xy)^2 \in Z(R)$ …………….2.14.
But by hypothesis $(xy)^2 - yx \in Z(R)$
Hence we have $yx \in Z(R)$ ………………….2.15.
Now again replacing x by $x+1$ in 2.15, we get
$yx + y \in Z(R)$. ……………………………2 16.
Using 2.15, and 2.16, we obtain $y \in Z(R)$ for all y∈ R, then R is commutative.
Theorem 2.4 If R is a 2-torsion free primitive ring which satisfy
$(xy)^2 - xy \in Z(R)$ for all x, y in R, then R is commutative.
Proof:- By hypotheis, $(xy)^2 - xy \in Z(R)$ . ... 2.17.
Replacing x by $x+y$ in 2.17, we obtain
$(xy)y^2 - y^2(xy) + y^4 - y^2 \in Z(R)$ ……………..2.18.
Now replacing x by y in $(xy)^2 - xy \in Z(R)$ we get
$y^4 - y^2 \in Z(R)$. …………….. 2.19.
Using 2.18, we obtain
$(xy)y^2 - y^2(xy) \in Z(R)$. ……….. 2.20.
We replacing x by $x+y$ in 2.20,
then $(xy)y^2 + y^4 + y^2(xy) + y^4 \in Z(R)$.
By 2.13 $y^4 + y^4$ ∈Z(R), i.e., $2y^4 \in Z(R)$.
Since R is a 2-torsion free ring,

$y^4 \in Z(R)$ ……….. 2.21.

Using 2.19, and 2.21, we obtain

$y^2 \in Z(R)$ ………….2.22.

Replacing y by xy in 2.22, we get $(xy)^2 \in Z(R)$
But by hypothesis $(xy)^2 - xy \in Z(R)$
Hence, $xy \in Z(R)$. …………..2.23.
Replacing y by $x+y$ in 2.22, weget $x^2 + y^2 + xy + yx \in Z(R)$.
Since $x^2, y^2 \in Z(R)$, we get
$xy + yx \in Z(R)$. ………………2.24.
From 2.23, and 2.24, $yx \in Z(R)$ Hence $xy - yx \in Z(R)$
If R is a primitive ring then R has a maximal right ideal which contains no non-zero ideal of R. Consequently, we obtain$(xy - yx)R = 0$, which fields $xy - yx = 0$ due to primitivity of R. Hence R is commutative.
Theorem 2.5 Let R be a 2-torsion free primitive ring which satisfy the identity
$(xy)^2 - yx \in Z(R)$ for all x, y in R. Then R is commutative.
Proof:- Given $(xy)^2 - yx \in Z(R)$ …………….. 2.25.
Replacing x by $x+y$ in 2.25, and using 2.25, we obtain
$(xy)y^2 + y^2(xy) + y^4 - y^2 \in Z(R)$ …….2.26.
Replacing x by y in 2.25, we get

$y^4 - y^2 \in Z(R)$ ………….. 2.27.

Using 2.26, and 2.27, we get

$(xy)y^2 + y^2(xy) \in Z(R)$ …………..2.28.
Now we replacing x by $x + y$ in 2.28,
then $(xy)y^2 + y^4 + y^2(xy) + y^4 \in Z(R)$.
But by 2.28, $y^4 + y^4 \in Z(R)$. i.e., $2y^4 \in Z(R)$.
Since R is a 2-torsion free ring, then $y^4 \in Z(R)$ .... 2.29.
Using 2.27, and 2.29, we get $y^2 \in Z(R)$ .... 2.30.
Now replacing y by xy in 2.30, $(xy)^2 \in Z(R)$.
By hypothesis, $(xy)^2 - yx \in Z(R)$,
Hence, $yx \in Z(R)$ .... 2.31.
Replacing y by $x + y$ in 2.31, we get

$xy \in Z(R)$. .... 2.32.

Hence $xy - yx \in Z(R)$.
Now using the same argument as in the proof of theorem 2. 4, we conclude that R is commutative.
Now we give some examples showing that unity in the statement of the theorems is essential.
Example 2.1:- The ring R = $\left\{\begin{bmatrix} a & b \\ 0 & o \end{bmatrix} / \ a, b \in Z_4\right\}$ is not of 4-torsion free and without unity. However it satisfies the identity (i) of theorem 2.1. Yet R is not commutative. This shows that conditions of being 4-torsion free and having unity are essential.
Example 2.2:- The ring R = $\left\{\begin{bmatrix} a & 0 \\ b & o \end{bmatrix} / \ a, b \in Z_4\right\}$is a counter example to (ii) of theorem 2.1 to establish that the conditions of being 4-torsion free and having unity are essential.
Example 2.3:- Let R =$\left\{\begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} \Big/ \ a, b, c \in Z\right\}$ Clearly, R is not commutative though it satisfy the relations $(xy)^2 - xy \in Z(R)$, or $(xy)^2 - yx \in Z(R)$, for all x, y in R.

References:-

1. Primitive alternative rings and semi - simplicity, Amer. J. Math., 77 (1955), 725-730.
2. On centers of alternative algebras, Comm., algebra 8 (1980), 289-297.
3. Anticommutative elements in alternative rings, J. of algebra 83, 65 -71 (1983).
4. On rings with alternators in the commutative center, Port. Math. 50 (1993), 297 - 304.
5. Right alternative rings, Proc. Amer. Math. Soc. 4(1953), 939 – 944
6. Alternative nil rings, Ann. of Math., 66 (1957), 395 - 399.