International Journal of Engineering, Science and Mathematics
Vol. 15 Issue 8, Aug 2026,
ISSN: 2320-0294 Impact Factor: 8.215
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

STUDY OF THE STURCTURE OF RINGS WITH (x, y, z) = (y, z, x)

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Abstract

We discuss the structure of rings satisfying the identity (x,y,z) = (y,z,x). If R is a ring satisfying the identity (x,y,z) = (y,z,x) and if it also satisfies the identity (x,x,x) = 0, then R is alternative. It is known that if R satisfies (x,y,z) = (y,z,x), it need not be an alternative ring [11]. Thus the class of rings satisfying this identity is a non-trivial extension of the class of alternative rings. Jordan remarked that $(x,x,x)^2 = 0$ is an identity in R. Outcalt strengthened this remark by proving that $(y,x,x)^2 = 0$ for all x,y ∈ R.

Keywords:- structure of rings, primitive ring, arbitrary ring, assosymmetric ring, Teichmuller identity, Semi-prime rings, Alernative Ring,

Introduction

We present this in section 1 to show that if R is a simple or primitive ring of char. ≠ 2, 3 satisfying (x,y,z) = (y,z,x), then R is alternative.

In section 2 it is shown that a ring R of the same characteristic satisfying (x,y,z) = (y,z,x) and containing no nonzero ideal whose square is zero must be alternative.

Kleinfeld [15] studied local versions of some well-known identities. In a similar way we consider a ring satisfying the identity h(x,x,y) = h(x,y,x) = h(y,x,x), where h(x,y,z) = (x,y,z) - (y,z,x).

In section 3 we show that this ring satisfies the identies (x,y,z) = (y,z,x) and ((y,x,x), x,x) = 0. This last identity gives a short step to prove that for a fixed element x and arbitrary y, the ideal generated by all (y,x,x) squares to zero. Then it is known that R is alternative.

1. Simple and Primitive rings with (x,y,z) = (y,z,x)

We consider a ring R of char. ≠ 2 or 3 satisfying the identity

$$(x, y, z) = (y, z, x) \qquad \text{......... 1.1}$$

for all x,y,z in R. In this section we present some results of Outcalt [22] on the structure of R. We prove that if R is either simple or primitive, then R is a Cayley - Dickson algebra or associative. These results are identical with those for alternative rings [9,10], Also that R is a Cayley-Dickson algebra or associative if R is simple, generalizes kleinfeld's theorem that an assosymmetric ring of char. ≠ 2 or 3 without ideals I ≠ 0 such that $I^2 = 0$ is associative [11]. This result is shared by many classes of associator dependent rings [18]. Associator dependent rings are defined to be rings satisfying the identity

$$(x, x, x) = 0 \qquad \text{.......... 1.2}$$

and $\alpha_1(x,y,z)+ \alpha_2(y,z,x)+ \alpha_3(z,x,y)+ \alpha_4(x,z,y)+ \alpha_5(z,y,x)+ \alpha_6(y,x,z) = 0$ -------1.3

for fixed α_i in some field of scalars [12]. A natural extension of the study of associator dependent rings is to drop the identity 1.2 and study rings satisfying a particular form of 1.3 which does not imply 1.2. Rings satisfying 1.1 are such a class of rings.

Now we study the properties of a function f (w,x,y,z) in R which is introduced in [3]. The function f(w,x,y,z) is defined by

$$f (w,x,y,z) = (wx,y,z) - x (w,y,z) - (x,y,z) w \qquad \text{--------1.4}$$

Then we use those properties to derive some useful identities. We need the Teichmuller identity, which holds in an arbitrary ring.

$0 = h (w,x,y,z) = (wx,y,z) - (w,xy,z) + (w,x,yz) - w (x,y,z) - (w,x,y)z.$

Now, expanding f and h and collecting terms yields

$f (w,x,y,z) - h(w,x,y,z) - h(x,y,z,w) = - (zw,x,y) + w (z,x,y) + (w,x,y)z$

But $f(z,w,x,y) = (zw,x,y) - w (z,x,y) - (w,x,y)z$.

Hence $f(w,x,y,z) = -f(z,w,x,y)$, --------1.5

Then $f(w,x,y,z) - f(z,w,x,y) + f(y,z,w,x) - h(y,z,w,x)$

$= (w,(x,y,z)) - (x,(y,z,w)) + (y, (z,w,x)) - (z,(w,x,y))$, by expanding f and h and collecting terms.

However, by 1.5 we have

$f(w,x,y,z) - f(z,w,x,y) + f(y,z,w,x) = 3f(w,x,y,z)$.

Hence

$3f(w,x,y,z) = (w,(x,y,z)) - (x,(y,z,w)) + (y,(z,w,x)) - (z,(w,x,y))$. 1.6

Letting w = x and y = z in 1.6, we obtain f (x,x,y,y) = 0, which yields, by linearizing,

$f(w,x,y,z) + f(x,w,y,z) + f(w,x,z,y) + f(x,w,z,y) = 0$. --------1.7

Using 1.5 and 1.7, we can write

$f(w,x,y,z) = f(x,w,z,y) + f(z,y,x,w) - f(w,x,z,y)- f(y,z,x,w) - 3f(x,w,z,y)$.

Expanding the first four items using 1.4 and the last using 1.6, we obtain

$f (w,x,y,z) = ((x,w),z,y) + (x,w,(z,y))$. --------1.8

Using 1.4 and 1.6 to expand 3f(x,y,x,z), we obtain, by subtracting the resulting expressions,

$A(x,y,z) = 3(xy,x,z) - x(y,x,z) - 2(y,x,z)x - 2y(x,x,z) - (x,x,z)y - (x, (x,y,z)) + (z, (y,x,x))= 0$.

Similarly, using 1.4 and then 1.6 to expand 3f (x,x,y,z), we obtain

$B(x,y,z) = 3(x2,y,z) - 3x(x,y,z) - 3(x,y,z)x - (y,(z,x,x)) + (z,(y,x,x)) = 0$.

Also, by 1.7 we have $0 = 2f(y,z,x,x) - f(z,y,x,x) - 3f(y,z,x,x)$.

Expanding the first two terms using 1.4 and then the last term using 1.6, we obtain

$$2(yz,x,x) - (zy,x,x) - z(y,x,x) - (z,x,x)y = 0. \qquad \text{--------1.9}$$

Next, we wish to establish the important identity

$$3((y,x,x), x,x) + ((x,x,x), x,y) + ((x,x,x), y,x) = 0. \qquad \text{--------1.10}$$

Indeed, if we expand f by 1.4, we observe that $f(x,x,x\backslash x) = f(x,x,x,x2)$;

Hence application of 1.5 yields

$$f (x2,x,x,x) = 0. \qquad \text{--------1.11}$$

We note that in an arbitrary ring, $(x,x,x) = (x2, x) = - (x, x2)$.

Thus $((x,x,x), x,x) = - ((x,x2), x,x)$;

But by 1.8 $\quad -((x,x2), x,x) = -f(x2,x,x,x)$.

Hence by 1.11 we have

$((x,x,x), x,x) = 0.$ --------1.12

Substituting x+y for x in 1.12 and then -x+y for x in 1.12 and adding yields

$3 ((x,x,y) x,x) + ((x,x,x), x,y) + ((x,x,x), y,x) + ((y,y,y), x,x) + 3((y,y,x), y,x)$
$+3((y,y,x), x,y) + 3((y,x,x), y,y) = 0,$ --------1.13

Since terms with one or three x's cancel, and terms with no x's or five x's go out by 1.12. Similarly, substituting x + 2y for x in 1.12 and then - x + 2y for x in 1.12 and adding yields

$3((x,x,y), x,x) + ((x,x,x),x,y) + ((x,x,x),y,x)+ 4 ((y,y,y),x,x) + 12 ((y,y,x),y,x) +$
$12 ((y,y,x), x,y) + 12 ((y,x,x),y,y) = 0.$

Subtracting 1.13 from this last equation and cancelling by 3, we obtain

$((y,y,y),x,x) + 3((y,y,x),y,x) + 3((y,y,x),x,y) + 3 ((y,x,x),y,y) = 0,$

which yields 1.10 on comparing with 1.13

Proceeding from 1.10 in the same way as we did from 1.12, substituting x + z for x in 1.10 and then x - z for x in 1.10 and adding yields

$0 = ((y,x,x),z,z) + ((y,x,z),z,x) + ((y,x,z),x,z) + ((y,z,x),z,x) + ((y,z,x),x,z) + ((y,z,z),x,x) + ((z,x,x),z,y) + ((z,z,x),x,y) + ((z,x,x),y,z) + ((z,z,x),y,x).$ ------- 1.14

Letting y = x in 1.14 yields

$0 = ((x,x,x),z,z) + 3((z,x,x),z,x) + 3((z,x,x),x,z) + 3((z,z,x),x,x)$ ------- 1.15

Similarly, we substitute x + y for x in 1.15 and then - x + y for x in 1.15 and add to obtain.

$0 = ((y,x,x),z,z) + ((z,x,x),z,y) + ((z,x,y), z,x) + ((z,y,x),z,x) + ((z,x,x),y,z) + ((z,x,y),x,z) + ((z,y,x), x,z) + ((z,z,x),x,y) + ((z,z,x),y,x) + ((z,z,y),x,x).$ --------1.16

Next, we wish to substitute (x,x,x) for y is 1.16. However, before doing that we shall introduce some new notation to make the equations less cumbersome to work with:

For $x \in R$, we define $v^n (a), a \in R$, by $v(a) = (a,x,x)$, $v^k (a) = v(v^{k-1} (a))$.

This $v^n (a)$ depends upon x as well as a.

Clearly, $v^m (v^n (a)) = v^n (v^m (a)) = v^{m+n} (a)$, and $v^n (a + b) = v^n (a) + v^n (b)$.

Equations 1.9, 1.10, and 1.12 become, respectively,

$C (y,z) = 2v (yz) - v(zy) - zv(y) - v(z)y = 0.$

$D (y) = 3 v^2 (y) + (v(x), x,y) + (v(x), y,x) = 0$, and $v^2(x) = 0$ 1.17

We shall derive some basic identities involving the v - function, and then we shall make the substitution y = v(x) in 1.16. This substitution will lead to the identity $v^4 (y) = 0$, which in turn yields the identity $v^3 (a) v^3 (b) = 0$.

We shall also show that if $a^2 = 0$ implies a = 0 in R, then R is alternative.

Now, expanding 0 = B (y,x,x) yields

$v(y^2) = yv(y) + v(y)y.$ -...... 1.18

Also, expanding

$0 = B (y, v(x), x) + B(y, x, v(x)) - 3D(y2) + 3D(y)y + 3yD(y)$ yields

$0 = -9v2(y2) + 9v2(y)y2 + 9yv2(y);$

Hence, by 1.18 we have

$V^2 (y^2) = v(yv(y) + v(y)y) = v^2 (y)y+ y v^2 (y).$ --------1.19

Finally, computing 0 = C (v(y), y) + C (y, v(y)), we obtain

$0 = v(v(y)y + \mathbf{yv(y)}) – yv^2 (y) – v^2 (y)y - 2(v(y))^2;$

Hence 1.19 yields

$(v(y))^2 = 0.$ 1.20

Equation 1.20 has far-reaching consequences. In the first place, Jordon's result that $(v(x))^2 = 0$, which we mentioned in introduction, is a special case of 1.20 and we have the following theorem.

Theorem 1.1: If $a^2 = 0$ implies $a = 0$ in R, then R is alternative.

Linearizing 1.20 yields $E(a,b) = v(a)(vb) + v(b)v(a) = 0$.

Since $(v(a), v(a), v(b)) = (v(a), v(b), v(a)) = (v(b), v(a), v(a))$,

Using 1.20 we obtain

$$3(v(a), v(a), v(b)) = -v(a).\, v(a)v(b) + v(a)v(b).\, v(a) - v(a).\, v(b)v(a) + v(b)v(a).\, v(a)$$
$$= -v(a)E(a,b) + E(a,b)v(a) = 0.$$

Hence $(v(a), v(a), v(b)) = 0$. --------1.21

Linearizing 1.21 we obtain

$$(v(a), v(b), v(c)) + (v(b), v(a), v(c)) = 0. \quad .... \ 1.22$$

Finally, by 1.22, $0 = v(b)E(a,c) - E(a,b)v(c) + (v(a), v(b), v(c)) + (v(b), v(a), v(c))$,

Which yields, by expansion,

$$v(a).\, v(b)v(c) = v(b).\, v(c)v(a). \quad \text{--------}1.23$$

Now, computing $0 = 2C(a,b) + C(b,a)$, we obtain

$$F(a,b) = 3v(ab) - av(b) - 2v(b)a - v(a)b - 2bv(a) = 0.$$

Next, we expand

$0 = 3v(F(a,b)) + F(a,v(b)) + 2F(v(b),a) + F(v(a),b) + 2F(b,v(a)) + 8E(a,b)$ to obtain

$G(a,b) = 9v^2(ab) - 5av^2(b) - 4v^2(b)a - 5v^2(a)b - 4bv^2(a) - 2v(a)v(b) = 0.$

Linearizing 1.19, we have

$H(a,b) = v^2(ab+ba) – av^2(b) – v^2(a)b – bv^2(a) – v^2(b)a = 0.$

Then, we compute $0 = G(a,b) - 4H(a,b)$ to obtain

$1(a,b) = 5v^2(ab) - 4v^2(ba) - av^2(b) - v^2(a)b - 2v(a)v(b) = 0.$

Finally, expanding

$0 = 3v(G(a,b)) + 5F(a, v^2(b)) + 4F(v^2(b),a) + 5F(v^2(a),b) + 4F(b, v^2(a)) + 2F(v(a), v(b)) + 15E(a, v(b)) + 15E(v(a),b)$ yields

$J(a,b) = 27v^3(ab) -13av^3(b)a -14v^3(b)a -13v^3(a)b-14bv^3(a)-3v^2(b)v(a)-3v(b)v^2(a) = 0.$

Letting $y = v(x)$ in 1.16, subtracting.

$0 = (D(z),x,z) + (D(z),z,x) + D((z,z,x))$, and applying 1.17, we obtain

$0 = (v(z),z,v(x)) + (v(z),v(x),z) + v((z,z,v(x))) - 3(v^2(z),x,z) - 3(v^2(z),z,x) -3v2((z,z,x))$. 1.24

Substituting $x + y$ for z in 1.24, subtracting $0 = D(v(y)) + v(D(y))$, and applying 1.17 yields

$$9v^3(y) = (v(x), v(x),y). \quad .... \ .1.25$$

We obtain $v^4(y) = 0$, -------- 1.26

by letting $y = v(y)$ in 1.25 and applying 1.21. Next, we expand

$0 = 3v(J(a,b)) + 13F(a, v^3(b)) + 14F(v^3(b), a) + 13F(v^3(a),b)+ 14F(b, v^3)(a)) + 3F(v^2(b),v(a)) + 3F(v(b), v^2(a)) + 43E(a, v^2(b))+ 43E(v^2(a),b) + 6E(v(a), v(b))$ to obtain, upon applying 1.26,

$2v(a)v^3(b) + 2v^3(a)v(b) + 3v^2(a)v^2(b) = 0.$ 1.27

Finally, letting $a = v(a)$ and $b = v(b)$ in 2.1.27 and applying 2.1.26, we obtain

$v^3(a)v^3(b) = 0.$ -------- 1.28

We define $U(x) = \{u \in R / u(r,x,x) = (r,x,x)u = (u,x,x) = 0$ for all $r \in R)$.

In terms of the v-function, this can be written as $uv(r) = v(r)u = v(u)=0$.

Lemma 1.1: U(x) is an ideal of R for all x in R.

Proof: Clearing U(x) is a subgroup of the additive group of R. Now,

$$v(ur) = v(ru) = 0 \qquad \text{.......... 1.29}$$

for $u \in v(x)$ and $r \in R$, since $3v(ur) = F(u,r) = 0 = F(r,u) = 3v(ru)$.

Let $u \in U(x)$ and $r,s \in R$. Then computing

$0 = uC(r, s) + C(r, s)u - (u, s, v(r)) - (u, v(s), r) + (s,v(r), u) + (v(s), r, u)$ yields

$$us.\,v(r) = -v(s).\,ru. \qquad \text{--------1.30}$$

Next, we obtain $us.\,v(r) = -2v(us.\,r)$ 1.31

by expanding $0 = C(r,u\,s) - C(r\,u,s) + 2v(r,u,s)) - 3v(u, s, r)) + v(s,r,u))$

and applying 1.29 and 1.30. Furthermore, expanding $0 = 2F(us, r)$

and applying 1.29 and 1.31, we have

$$5\,us.\,v(r) = -4v(r).\,us. \qquad \text{.......... 1.32}$$

We establish that

$$us.\,v(r) = -4ru.\,v(s), \qquad \text{--------1.33}$$

by expanding $0 = 4C(r,s)u + 4(s, v(r), u) + 4(v(s), r,u) - 4(v(r), u,s) - 4(r, u, v(s))$

and applying 1.30 and 1.32. Finally, expanding $0 = 4F(ru, s) - 4F(r,us) - 12v((r,u,s)) + 12v(u, s, r))$

and applying equations 1.29 through 1.33, we obtain $6us.v(r) = 0$. This latter, along with 1.29, 1.30, 1.32 and 1.33, completes the proof.

Now R will be said to have property P providing there is a subgroup N of the additive group of R such that $N + I = R$ for every ideal $I \neq 0$ of R. Property P is of interest because clearly R has property P if R is simple (take N= 0), and we shall see that R has property P if R is primitive.

Lemma 1.2 : If R has property P, then $v(R) \subset U(x)$ for all x in R.

Proof: Letting $a = x$ and $b = v^2(a)$ in 1.21, we have $(v(x), v(x), v^3(a)) = 0$, and by 1.25 and 1.28,

$$V^3(a)\,(v(x), v(x), r) = 0 = (v(x), (v(x), r)\,v^3(a);$$

hence $v^3(a) \in U(v(x))$. If $U(v(x)) = 0$, then $v^3(a) = 0$;

if $U(v(x)0 \neq 0$, then $N + U(v(x)) = R$. For $n \in N$, $u \in U(v(x))$,

we have, using 1.25, $9v^3(n+u) = 9v^3(n) + (v(x), v(x), u) = 0$.

Hence, in either case we have

$$v^3(a) = 0. \qquad \text{.......... 1.34}$$

Applying 1.34 to $J(a,b) = 0$ yields

$$v^2(b)\,v(a) + v(b)\,v^2(a) = 0, \qquad \text{--------1.35}$$

and we obtain

$$v^2(b)\,v^2(a) = 0 \qquad \text{--------1.36}$$

if we let $a = v(a)$ in 1.35 and apply 1.34.

The next sequence of identities will show that $v^2(b) \in U(v^2(a))$.

Expanding $0 = B(v^2(a), v^2(a),r)$ and applying 1.20 and 1.21 with $a = b = v(a)$ yields

$$2v^2(a)\,(v^2(a), v^2(a),r) + (v^2(a), v^2(a), r)\,v^2(a) = 0. \qquad \text{.......... 1.37}$$

Similarly, expanding $0 = B(v^2(a), r,v^2(a))$ yields

$$-2v^2(a)\,(v^2(a), v^2(a), r) - 4(v^2(a), v^2(a), r)\,v^2(a) = 0. \qquad \text{.......... 1.38}$$

By adding 2.1.37 to 2.1.38 we obtain

$$v^2(a)\,(v^2(a), v^2(a),r) = 0 = (v^2(a), v^2(a),r)\,v^2(a) \qquad \text{--------1.39}$$

by referring to 1.38. Substituting a + b for a in 1.39 and then - a + b for a in 1.39 and adding yields

$v^2(a)\,[(v^2(a), v^2(b), r) + (v^2(b), v^2(a), r)] = -\,v^2(b)\,(v^2(a), v^2(a),r)$, --------1.40

$[(v^2(a), v^2(b),r) + (v^2(b), v^2(a),r)]\,v^2(a) = -(v^2(a), v^2(a),r)\,v^2(b)$

since terms with one or three a's cancel, and terms with no a's go out by 1.39.

Now, applying 1.20 to the expansion of $0 = B(v^2(a), v^2(b),r) + B(v^2(a),r, v^2(b))$ yields

$0 = -3\,v^2(a)\,[(v^2(a), v^2(b),r) + (v^2(b), v^2(a),r)] - 3[(v^2(a), v^2(b), r) + (v^2(b), v^2(a),r)]\,v^2(a)$,

to which we apply 1.40 to obtain

$v^2(b)\,(v^2(a), v^2(a),r) + (v^2(a), v^2(a),r)\,v^2(b) = 0.$ --------1.41

Furthermore, computing

$0 = A\,(v2(a), r, v2\,(b)) + B(v2\,(a), r, v2\,(b)) - 3\,h(v2\,(a), v2(b), v2(a), r)$

and applying 1.20, 1.21,1.36 and 1.40, we have $0 = 3\,v^2(b)\,(v^2(a), v^2(a), r)$.

Hence, referring to 1.41, we see that

$v^2(b)\,(v^2(a), v^2(a), r) = 0 = (v^2(a), v^2(a), r)\,v^2(b).$ --------1.42

Equations 1.21 and 1.42 show that $v^2(b) \in U(v^2(a))$.

If $U(v^2(a)) = 0$, then $v^2(b)=0$; if $U(v^2(a)) \neq 0$, then $N + U(v^2(a)) = R$. For $n \in N$, $u \in U(v^2(a))$, we have, using 1.25 with $x = v(a)$, $(n+u, v^2(a).\,v^2(a)) = (n, v^2(a), v^2(a))$

$= 9(((n, v\,(a), v(a)), v\,(a), v(a)), v(a), v(a)) = 0$

since $((n, v(a), v(a)), v(a), v(a)) = 0$ by the definition of N.

Hence, in either case we have

$$(v^2(a), v^2(a), r) = 0. \qquad \text{--------1.43}$$

Evaluating the associator in 1.43 and applying 1.20, we obtain

$$v^2(a).\,v^2(a)r = 0, \qquad \text{--------1.44}$$

and linearizing 1.43 yields

$0 = (v^2(a), v^2(b), r) + (v^2(b), v^2(a), r).$ --------1.45

Since by 1.45 we have $0 = I\,(a,a)\,v^2(r) - v^2(a)\,I(a, r) + (v^2(a), a, v^2(r)) + (a, v^2(a), v^2(r))$,

We obtain, by expanding and applying 1.20, 1.36, and 1.44, $0 = 2v^2(a).\,v(a)\,v(r)$;

But then, using 1.23 with $a = r$, $b = v(a)$ and $c=a$ yields

$$0 = v\,(r).\,v^2(a)\,v(a). \qquad \text{--------1.46}$$

It is easy to see that 1.1 holds in the anti-ismorphic copy R1 of R and that R1 has property P if R does; Hence 2.1.46 holds in R1. But then we have in R the identity $0 = v(a)\,v2(a).\,v(r)$, from which we subtract $0 = E(a, v^2(a))\,v(r)$ to obtain

$$0 = v^2(a)\,v(a).\,v(r). \qquad \text{.......... 1.47}$$

Computing $0 = F(v^2(a)\,v(a))$ and applying 1.20 and 1.34 yields

$$v\,(v^2(a)\,v(a)) = 0. \qquad \text{-------1.48}$$

Equations 1.46, 1.47 and 1.48 imply that $v^2(a)\,v(a) \in U(x)$.

If $U(x) = 0$, then $v^2(a)\,v(a) = 0$; if $U(x) \neq 0$, then $N + U(x) = R$. For $n \in N$, $u \in U(x)$, we have $v\,(n+u) = 0$. Hence, in either case we have

$$v^2(a)v(a) = 0. \qquad \text{--------1.49}$$

Linearizing 1.49 yields $v^2(a)\,v(b) + v^2(b)\,v(a) = 0$, from which we subtract 1.35 to obtain

$0 = v^2(a)\,v(b) - v(b)\,v^2(a) + E(b, v(a)) = 2v^2(a)\,v(b)$;

Hence $v^2(a)\,v(b) = v(b)\,v^2(a) = 0.$ 1.50

Equations 1.34 and 1.50 imply that $v^2(a) \in U(x)$. If $U(x) = 0$, then $v^2(a)=0$; if $U(x) \neq 0$, then $N + U(x) = R$. For $n \in N$, $u \in U(x)$, we have as before $v^2(n+u) = 0$.
Hence, in either case we have $v^2(a) = 0$. But then computing $0 = G(a, b)$ yields $v(a) v(b) = 0 = v(b) v(a)$. Therefore, $v(a) \in U(x)$.
This completes the proof of the lemma.
Now we prove the structure theorems of R.
Theorem 1.2: A simple ring R of char. $\neq 2$ or 3 satisfying 1.1 is alternative and hence a Cayley-Dickson algebra or associative.
Proof: Clearly R has property P if R is simple. Hence, by lemma 1.2, $v(a) \in U(x)$, which implies that $v(a) = 0$ if $U(x) = 0$, or if $U(x) = R$. Therefore, R is alternative.
From Kleinfeld's [10] theorem, it follows that R is a Cayley-Dickson algebra or associative.
Theorem 1.3: A primitive ring R of char. $\neq 2$ or 3 satisfying 1.1 is alternative and hence a Cayley - Dickson algebra or associative.
Proof: First we shall show that R has property P if R is primitive. Let M be a maximal right ideal of R which contains no ideals $I \neq 0$ of R.. The letter m will denote an element of M; other letters will denote arbitrary elements of R. By $r \equiv s$ we mean that $r-s \in M$.
Clearly $(m, x, y) \equiv 0$, -------- 1.51
since M is a right ideal. In particular, it follows from 1.51 that $v(m) = 0$. Also, since M is a right ideal, 1.51 yields

$$v(my) \equiv v(m)y \equiv 0. \qquad \text{--------1.52}$$

Expanding $0 = F(y,m) - C(m,y)$ and applying 1.52 we have

$$v(ym) \equiv 0. \qquad \text{-------1.53}$$

Furthermore, expanding $0 = F(y,m)$ and applying both 1.52 and 1.53 yields

$$yv(m) + v(y)m \equiv 0. \qquad \text{--------1.54}$$

Next, we expand $0 = E(m, y)$ and apply 1.52 to obtain

$$v(y)\, v(m) \equiv 0. \qquad \text{--------1.55}$$

Letting $m = v(m)$ in 1.54, we have by 1.55

$$yv^2(m)=0. \qquad \text{------- 1.56}$$

Now we define $S = \{s \in M / Rs \equiv 0\}$.
S is an ideal of R. Indeed, for $s \in S$, $a, b \in R$, 1.51 yields $a. sb \equiv as. b \equiv 0$ and $a.bs \equiv ab. s \equiv 0$. Hence, $S = 0$ since $S \subset M$. But then $v^2(m) = 0$, since $v^2(m) \in S$ by 1.56. Hence r has property P since $M + I = R$ for all ideals $I \neq 0$ of R. Therefore, $v(a) \in U(x)$ by lemma 1.2. If $U(x) \subset M$, then, $U(x) = 0$ and $v(a) = 0$; if $U(x) \not\subset M$, then $M + U(x) = R$. For $m \in M$, $u \in U(x)$, $v(u+m) = v(m)$. Hence in either case, 1.51 yields

$$v(a) \equiv 0. \qquad \text{.......... 1.57}$$

Computing $0 = C(a,b)$ and applying 1.57 yields $bv(a) \equiv 0$. Thus $v(a) \in S$ which implies $v(a) = 0$. Hence R is alternative and we can apply Theorem 2 of [9] to complete the proof.

We give an example of a ring satisfying the identity $(x,y,z) = (y,z,x)$, which is not alternative.

Example: Consider the algebra having basis elements x,y,z over an arbitrary field. We define $x^2 = y$, $yx = z$, and all other products of basis elements equal to zero. Then all associators involving basis elements are zero except $(x,x,x) = z$. Thus it satisfies the identity $(x,y,z) = (y,z,x)$. But it is not alternative.

2. Semi-prime rings with (x,y.z) = (y.z.x)

In section 1, we have seen that if R is a simple or primitive ring satisfying (x,y,z) = (y,z,x) with char. ≠ 2, 3, then R is alternative and hence associative or a Cayley-Dickson algebra. In this section we extend these results and prove that a ring R satisfying (x,y,z) = (y,z,x) with the same restriction on its characteristic and containing no non-zero ideal whose square is zero is alternative. The structure theorems of section 1 are derived using the ideal U(x) and a property P, which is shared by both simple and primitive rings. It is not obvious that this property P holds for a ring having no nonzero ideal whose square is zero. Here we present three lemmas due to Sterling [27]. Using these lemmas and Outcalt's [22] argument, we prove the main result.

Throughout this section R denotes a ring satisfying the identity

$$(x,y,z) = (y,z,x), \qquad \text{.......... 2.1}$$

with no nonzero ideal whose square is zero and with char. ≠ 2, 3. We know that a ring is semiprime if it has no nonzero ideal whose square is zero.

In section 1, we have defined the v-function and obtained that

$$(v(a))^2 = 0 \qquad \text{--------2.2}$$

and v(a) (v(b) v(c)) = v(b) (v(c) v(a)). --------2.3

Also we have derived the identities F(a,b), G(a,b), I(a,b), J(a,b) and

$$9v^3(a) = (v(x), v(x), a), \qquad \text{---------2.4}$$

$$f(w,x,y,z) = -f(z,w,x,y). \qquad \text{---------2.5}$$

From lemma 1.1 we have U(x) = {u ∈ R / u (R,x,x) =(R,x,x)u = (u,x,x) = 0} is an ideal of R.

We use the following identity which is satisfied in an arbitrary ring:

C(x,y,z) = (xy,z) - x(y,z) - (x,z)y - (x,y,z) + (x,z,y) - (z,x,y) = 0.

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