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SEHGAL-GUSEMAN-TYPE FIXED POINT THEOREM IN B-RECTANGULAR METRIC SPACES

Asmita Yadav[a] and Dr. A.S. Saluja[b]

Department of Mathematics, Institute for Excellence in Higher Education (IEHE), Bhopal - 462016,Madhya Pradesh, India.
Email:asmitay201@gmail.com

ABSTRACT: In this paper, a Sehgal-Guseman-type fixed point theorem is proven in b-rectangular metric spaces. This provides a complete solution to an open problem posed by Zoran D. Mitrovic regarding a Banach`s fixed point theorem in b-rectangular metric space and b-metric spaces. the result generalizes and unifies certain findings in fixed point theory.
Keywords: fixed point; b-metric space; rectangular metric space ;b-rectangular metric space.

1. INTRODUCTION

Fixed point theory plays a crucial role in non linear functional analysis and applied mathematics. Since the publication of the Banach contraction principle, numerous researchers have expended and generalized it. Sehgal[1] contributed to this area by exploring fixed points for mappings with contractive iterates.

Theorem 1. Let (X, d) be a metric space and let T:X→ X be a continuous mapping which satisfies the condition that there exists a real number k,$0 \leq k < 1$ such that for each x there exists a positive integer $l(x)$ such that, for each $y \in X$,

$$d\left(T^{l(x)}x, T^{l(x)}y\right) \leq kd(x, y)$$

Then ,T has a unique fixed point.

Later, Guseman [2], Matkowsk [3] and others [4] discussed it in depth.

Many researchers have explored the Banach contraction principle with in various generalized metric spaces.

For instance, Branciari introduced rectangular metric spaces and established a Banach contraction principle. Bakhtin developed b-metric spaces. In 2015, George et al. introduced b-rectangular metric spaces, generalizing both rectangular and b-metric spaces. They proved analogues of the Banach contraction principle and Kannan`s fixed point theorem.

In 2018, Mitrovic relaxed the contraction coefficient in the Banach contraction principle for b-rectangular metric spaces from $k \in \left(0, \frac{1}{s}\right)$ to $k \in (0,1)$.

This paper proves the Sehgal -Guseman-type theorem in b-rectangular metric spaces, answering Mitrovic`s question. The result generalizes and unifies findings in fixed point theory.

2. PRELIMINARIES

Definition 2.1.([6,11]) Let X be a non-empty set, $s \geq 1$ be a given real number and let $d: X \times X \to [0, \infty)$ be a mapping, such that for all $x, y, z \in X$, the following conditions hold :

(b1) $d(x, y) = 0 \Leftrightarrow x = y$

(b2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ (b-triangular in equality)

Then the pair (X, d) is called a b-metric space.

Definition 2.2.([5]) let X be a non empty set, and let $d: X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from x *and* y :

(r1) $d(x, y) = 0$ if and only if $x = y$;

(r2) $d(x, y) = d(y, x)$;

(r3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality)

Then (X, d) is called a rectangular metric space or generalized metric space.

Definition 2.3.([7]) let X be a non empty set, $s \geq 1$ be a given real number and let $d: X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from x *and* y

(rb1) $d(x, y) = 0$ if and only if $x = y$

(rb2) $d(x, y) = d(y, x)$;

(rb3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ (b-rectangular inequality)

Then ,(X,d) is called a b-rectangular metric space or b-generalized metric space.

From the above definitions it is clear that every metric space is a rectangular metric space and a b-metric space. Also every rectangular metric space or every b-metric space is a b-rectangular metric space. Converse is not necessarily true.

3. MAIN RESULT

Theorem 3.1:let (X,d) be a b-rectangular metric space with coefficient $s > 2$ and let $T: X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < \frac{1}{2}$ such that for each x there exists a positive integer $p(x)$ such that , for each $y \in X$,

$$d\left(T^{p(x)}x, T^{p(x)}y\right) \leq kd(x, y)$$

Then , T has a unique fixed point.

Proof : Firstly we prove the theorem in case when $0 \leq k < \frac{1}{s}$

Let x_0 be an arbitrary point in X. Consider a sequence $\{x_n\}$ by

$$x_1 = T^{p(x_0)}x_0$$

$$x_2 = T^{p(x_1)}x_1$$

.

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.

$$x_{n+1} = T^{p(x_n)}x_n$$

If $x_{n_0} = x_{n_0+1}$for some $n_0 \in N$, then x_{n_0} is a fixed point of T.

Since $x_{n_0} = T^{p(x_{n_0})}x_{n_0}, x_{n_0}$ is a fixed point of $T^{p(x_{n_0})}$

To prove that x_{n_0} is a fixed point of T, we firstly prove that $x_{n_0}$is the unique fixed point of $T^{p(x_{n_0})}$.

$T^{p(x_{n_0})}v = v$ for some $v \neq x_{n_0}$, then $d(x_{n_0}, v) = d\left(T^{p(x_{n_0})}x_{n_0}, T^{p(x_{n_0})}v\right)$

$$\leq kd(x_{n_0}, v)$$

$$d(x_{n_0}, v) \leq kd(x_{n_0}, v)$$

Which is a contradiction, since $0 \leq k < \frac{1}{2}$

$$\text{Now } Tx_{n_0} = TT^{p(x_{n_0})}x_{n_0}$$

$$= T^{p(x_{n_0})}Tx_{n_0}$$

That is, Tx_{n_0} is also a fixed point of $T^{p(x_{n_0})}$.

By the uniqueness of fixed point of $T^{p(x_{n_0})}$, we have $Tx_{n_0} = x_{n_0}$, which shows that x_{n_0} is a fixed point of

In what follows, we suppose that $x_n \neq x_{n+1} \forall\, n \in N$.

Step-2:

Now we suppose that $x_m \neq x_n$ for $m \neq n$.

Without any loss of generality, suppose $m > n$.

If $x_m = x_n$ for $m \neq n$ then $d(x_m, x_{m+1}) = d\left(T^{p(x_{m-1})}x_{m-1}, T^{p(x_m)}T^{p(x_{m-1})}x_{m-1}\right)$

$$\begin{aligned}
&= d\left(T^{p(x_{m-1})}x_{m-1}, T^{p(x_{m-1})}x_{m-1}\right)\\
&\leq kd(x_{m-1}, T^{p(x_m)}x_{m-1})\\
&= kd\left(T^{p(x_{m-2})}x_{m-2}, T^{p(x_m)}T^{p(x_{m-2})}x_{m-2}\right)\\
&= k^2 d\left(x_{m-2}, T^{p(x_m)}x_{m-2}\right)\\
&\leq \cdots\\
&\leq k^{m-n} d\left(x_n, T^{p(x_m)}x_n\right)\\
&= k^{m-n} d\left(x_n, T^{p(x_m)}x_m\right)\\
&= k^{m-n} d(x_m, x_{m+1})
\end{aligned}$$

Which is a contradiction since $0 \leq k < \frac{1}{2}$.

Step 3: For $x \in X, q(x) = sup\, n\, d(T^n x, x)$ is finite.

Let $x \in X$ and let $h(x) = \{d(T^k x, x) \colon k = 1,2 \dots, p(x), p(x)+1, \dots 2p(x)\}$

If n is a positive integer, then there exists an integer $\alpha \geq 0$ such that

$$\alpha p(x) < n \leq (\alpha+1)p(x).$$

We can assume that $T^n x, T^{p(x)}x, T^{2p(x)}x, x$ are different from each other. Otherwise the conclusion is

$$\begin{aligned}
d(T^n x, x) &\leq s[d(T^n x, T^{p(x)}x) + d\left(T^{p(x)}x, T^{2p(x)}x\right) + d(T^{2p(x)}x, x)]\\
&\leq s[kd\left(T^{n-p(x)}x, x\right) + kd\left(x, T^{p(x)}x\right) + d(T^{2p(x)}x, x)\\
&\leq skd\left(T^{n-p(x)}x, x\right) + skz(x) + sz(x)\\
&\leq s^2 k\left[d\left(T^{n-p(x)}x, T^{p(x)}x\right) + d\left(T^{p(x)}x, T^{2p(x)}x\right) + d\left(T^{2p(x)}x, x\right)\right] + skz(x) + sz(x)\\
&\leq s^2k^2 d\left(t^{n-2p(x)}x, x\right) + s^2k^2 d\left(x, T^{p(x)}x\right) + s^2kd\left(T^{2p(x)}x, x\right) + skz(x) + sz(x)\\
&\leq s^2k^2 d\left(t^{n-2p(x)}x, x\right) + s^2k^2 z(x) + s^2kz(x) + skz(x) + sz(x)\\
&\leq \cdots \quad \dots\\
&\leq s^\alpha k^\alpha d\left(t^{n-\alpha p(x)}x, x\right) + (sz(x) + s^2kz(x) + \cdots)\\
&\quad + (skz(x) + s^2k^2z(x) + s^3k^3z(x) + \cdots)\\
&\leq s^\alpha k^\alpha z(x) + \frac{sz(x)}{1-sk} + \frac{skz(x)}{1-sk}\\
&\leq z(x) + \frac{sz(x)}{1-sk} + \frac{skz(x)}{1-sk}
\end{aligned}$$

Hence $q(x) = sup\, n\, d(T^n x, x)$ is finite.

Step 4: $d(x_n, x_{n+1}) = 0$

$$\begin{aligned} d(x_n, x_{n+1}) &= d\big(T^{p(x_{n-1})}x_{n-1}, T^{p(x_n)}T^{p(x_{n-1})}x_{n-1}\big) \\ &= d(T^{p(x_{n-1})}x_{n-1}, T^{p(x_{n-1})}T^{p(x_n)}x_{n-1}) \\ &\le kd\big(x_{n-1}, T^{p(x_n)}x_{n-1}\big) \\ &\dots \quad \dots \\ &\le k^n d\big(x_0, T^{p(x_n)}x_0\big) \\ &\le k^n\, qx_0 \end{aligned}$$

Then $d(x_n, x_{n+1}) = 0$

Step 5: $\{x_n\}$ is a Cauchy sequence in X.

For the sequence $\{x_n\}$, we consider $d(x_n, x_{n+l})$ in two cases. For the sake of convenience, we denote $q(x_0)$ by q_0.

If l is odd say $2m + 1$ then by step 2 and (rb3)

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\le s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\ &\le sk^n q_0 + sk^{n+1}q_0 + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\ &\le \dots \quad \dots \\ &\le sk^n q_0 + sk^{n+1}q_0 + s^2k^{n+2}q_0 + s^2k^{n+3}q_0 + s^3k^{n+4}q_0 + s^3k^{n+5}q_0 + \dots + s^m k^{n+m}q_0 \\ &\le sk^n q_0[1 + sk^2 + s^2k^4 + \cdots] + sk^{n+1}q_0[1 + sk^2 + s^2k^2 + \cdots] \\ &\le sk^n q_0\left(\frac{1}{1 - sk^2}\right) + sk^{n+1}q_0 \times \left(\frac{1}{1 - sk^2}\right) \\ &\le sk^n q_0\left[\frac{1}{1 - sk^2} + \frac{k}{1 - sk^2}\right] \\ &\le \frac{1 + k}{1 - sk^2}\; sk^n q_0 \end{aligned}$$

If l is even, say $2m$, then by step 2 and (rb3)

$$\begin{aligned} d(x_n, x_{n+2m}) &\le s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\ &\le sk^n q_0 + sk^{n+1}q_0 + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\ &\le \dots \quad \dots \\ &\le sk^n q_0 + sk^{n+1}q_0 + s^2k^{n+2}q_0 + s^2k^{n+3}q_0 + \dots + s^{m-1}k^{n+2m-4}q_0 \\ &\quad + s^{m-1}k^{n+2m-3}q_0 + s^{m-1}k^{n+2m-2}q_0 d(x_0, T^{n+2m-1}T^{n+2m-2}x_0) \\ &\le sk^n q_0[1 + sk^2 + s^2k^2 + \cdots] + sk^{n+1}q_0[1 + sk^2 + s^2k^2 + \cdots] + s^{m-1}k^{n+2m-2}q_0 \\ &\le \frac{1 + k}{1 - sk^2}\; sk^n q_0 + (sk)^{2m}k^{n-2}q_0 \\ &\le \frac{1 + k}{1 - sk^2}\; sk^n q_0 + k^{n-2}q_0 \end{aligned}$$

Then it follows from above argument

$$d(x_n, x_{n+l}) = 0 \text{ for all } l > 0.$$

Thus sequence $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, there exist a point $u \in X$ such that $x_n = u$

Step 6:

u is a fixed point of T.

By(3.1), $d\left(T^{p(u)}u, T^{p(u)}x_n\right) \leq d(u, x_n)$, then $d\left(T^{p(u)}u, T^{p(u)}x_n\right) = 0$

$$\begin{aligned} d\left(T^{p(u)}x_n, x_n\right) &= d\left(T^{p(u)}T^{p(x_{n-1})}x_{n-1}, T^{p(x_{n-1})}x_{n-1}\right) \\ &= d(T^{p(x_{n-1})}x_{n-1}, T^{p(x_{n-1})}T^{p(u)}x_{n-1}) \\ &\leq kd\left(x_{n-1}, T^{p(u)}x_{n-1}\right) \\ &\leq \cdots \quad \ldots \\ &\leq k^n d\left(x_0, T^{p(u)}x_0\right) \end{aligned}$$

That is $\lim_{n\to\infty} d\left(T^{p(u)}x_n, x_n\right) = 0$

By (rb3),

$$d\left(T^{p(u)}u, x_{n+1}\right) \leq s\left(d\left(T^{p(u)}u, T^{p(u)}x_n\right) + d\left(T^{p(u)}x_n, x_n\right) + d(x_n, x_{n+1})\right)$$

Then $\lim_{n\to\infty} d\left(T^{p(u)}u, x_{n+1}\right) = 0$

Therefore by (rb3), we have

$$d\left(u, T^{p(u)}u\right) \leq s\left(d(u, x_n) + d(x_n, x_{n+1}) + d\left(x_{n+1}, T^{p(u)}u\right)\right)$$

Putting limit $n \to \infty$ in the above inequality, we have

$$d\left(u, T^{p(u)}u\right) = 0$$

This means that $T^{p(u)}u = u$; that is u is a fixed point of $T^{p(u)}$.

Now, $d(u, Tu) = d\left(T^{p(u)}u, TT^{p(u)}u\right)$

$$\begin{aligned} &= d\left(T^{p(u)}u, T^{p(u)}Tu\right) \\ &\leq kd(u, Tu) \end{aligned}$$

Then $d(u, Tu) = 0$, that is u is a fixed point of T.

Step 7: u is the unique fixed point of T.

To prove that u is the unique fixed point of $T^{p(u)}$.

Let $T^{p(u)}v = v$ for some $v \neq u$, then

$$\begin{aligned} d(u, v) &= d\left(T^{p(u)}, T^{p(u)}v\right) \\ &\leq kd(u, v) \end{aligned}$$

Which is a contradiction. Since $0 \leq k < \frac{1}{2}$

If w is another fixed point of T, then $w = Tw = T^2w = \cdots = T^{p(u)}w$

w is a fixed point of $T^{p(u)}$ too.

By the uniqueness of fixed point of $T^{p(u)}$, we have $u = w$.

Therefore T has a unique fixed point.

4. Conclusion

This paper establishes a Sehgal -Guseman -type fixed point theorem in b-rectangular metric spaces , resolving an open question posed by Mitrovic. The findings presented here broaden and consolidate existing result in fixed point theory.

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