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NUMERICAL STUDY OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

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ABSTRACT

Singularly Perturbed Differential Equations (SPDEs) arise in various scientific and engineering applications, including fluid dynamics, quantum mechanics, and reaction-diffusion processes. These equations involve a small perturbation parameter leading to boundary layers, challenging numerical solutions. Classical numerical methods often fail to provide accurate results for such problems, necessitating specialized approaches. This paper investigates the mathematical solution of singularly perturbed boundary value problems (SPBVPs) using Laplace Transform Methods (LTM), including Euler's Method (EM) and Weeks Method (WM). The study formulates the problem, discusses the application of Laplace transformation, and explores numerical inversion techniques for obtaining solutions in the time domain. Numerical experiments are conducted to validate the proposed methods, demonstrating their effectiveness and accuracy. Outcomes are presented for different values of the perturbation parameter, showing that both EM and WM achieve high precision and stability. A comparative analysis with existing numerical schemes highlights the superior performance of the proposed methods in terms of convergence and absolute error reduction. The findings confirm that the suggested approaches provide reliable and efficient solutions for SPBVPs, making them suitable for practical applications in engineering and applied mathematics.

KEYWORDS: *SPBVPs, Laplace Transform Methods, EM, Weeks Method*

INTRODUCTION

Functional Differential Equations (FDEs) including forward and delayed arguments are often described as mixed-type FDEs. Problems involving optimal control served as the major source of motivation for the investigation of these equations [1]. A wide variety of practical applications could be found for these FDEs, including the theory of nerve conduction [4], the theory of economic dynamics [2], and the theory of traveling waves in a spatial lattice [3].

Singularly Perturbed Differential Equations (SPDEs) are a set of differential equations where the maximum derivative is multiplied by a tiny parameter [6]. Layers that occur in restricted portions of the domain are a common feature of solutions to these types of equations [7]. Fluid dynamics, elasticity, chemical reactions, the theory of "gas porous electrodes, the Navier-Stokes equations of fluid flow at high Reynolds number, oceanography, meteorology, reaction-diffusion processes, and quantum mechanics" are just a few examples of domains where this type of problem frequently arises in applied mathematics and physics [8]. The fact that these issues are influenced by a small positive parameter ε in such a manner that the solution displays a multiscale nature is widely recognized. In other words, there are thin transition layers where the solutions change quickly for small values of ε, but outside of these layers, it acts regularly and changes slowly. Thus, the existence of tiny parameters in singularly perturbed problems causes serious issues that must be handled to achieve accurate numerical solutions [9]. Figure 1 show the applications of SPBVP.

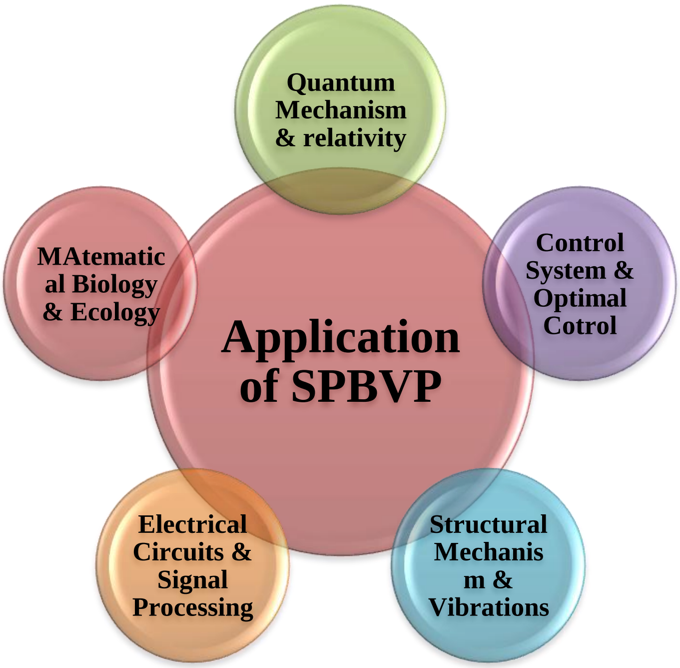

Figure 1: Application of SPPBVP

It is well recognized that traditional numerical techniques cannot provide good numerical solutions for singly perturbed situations [10,11]. It is crucial to create appropriate numerical techniques that converge uniformly concerning ε to solve these issues, indicating that their correctness is independent of the value of the parameter ε. To address such difficulties the research analyses the numerical solution to the SPBVPs utilizing the Laplace Transform Techniques (LTM) [12–14].

One of the most effective methods for solving integer and non-integer order linear differential equations is the Laplace transform (LT) [15]. As previously stated by:

$$F(s) = L(f(t)) = \int_0^{\infty} e^{-st} f(t)dt \qquad ; t \geq 0 \qquad (1)$$

One drawback of using the LT to solve differential equations is that it might provide Laplace domain solutions that aren't always analytically invertible to the time domain [16]. Numerical inversion methods are used to transform the solution derived from the Laplace domain into the time domain. Every approach has its uses and is best suited for a certain kind of issue. The Fourier series approach, the de Hoog method, the Steh-fest method, Talbot's method, and others are famous numerical approximations of LT. The research focuses on second-order SPDEs of the type that has been studied using the LT, Euler's Method (EM), and Weeks Method (WM).

$$\in \delta_1 \frac{d^2u(\tau)}{d\tau^2} + \delta_2 \frac{du(\tau)}{d\tau} + \delta_3 u(\tau), \quad \tau \in \Omega,$$

(2)

Another way to describe it is as

$$\mathcal{L}u(\tau) = f(\tau),$$

(3)

Under Dirichlet's boundary conditions, $\mathcal{L}$ is equal to $\epsilon\delta_1 \frac{d^2}{d\tau^2} + \delta_2 \frac{d}{d\tau} + \delta_3$.

$$\mathcal{B}u(\tau) = g_1(\tau), \quad \tau \in \partial\Omega,$$

(4)

In addition, initial condition

$$u(0) = \phi_1, \; u'(0) = \phi_2,$$

(5)

Where δ_1, δ_2, and δ_3 are the coefficients, and $f(\tau), g_1(\tau)$, and $u(0)$ are provided with differentiable functions that are suitably continuous. The boundary layer will be located on the left side of the domain if $\delta_2 \leq q_1 < 0$ and on the right end of the domain if $q_1 > 0$, because we also assume that $\delta_2 > q_1$ over the whole domain Ω. The solutions of SPBVPs exhibit a multi-scale nature, characterized by thin transition layers where the solution varies rapidly, referred to as the inner region, and areas where the solution changes gradually, known as the outer region.

LITERATURE REVIEW

There have been a great number of scholars who have investigated the numerical solution of SPDEs.

Kumari et al., (2025) [17] examined a network of time-dependent SPDEs with tiny changes, which are of special interest in the field of neurology. The approximation method of Taylor series expansions is used to control the delay and advance parameters of the equations. In the time direction, they discretize the problem using the "Crank-Nicolson finite difference technique" on a uniform mesh; in the space direction, they utilize a combination of a Shishkin-type mesh and the cubic B-spline collocation approach. Its efficiency and efficacy in actual applications have been confirmed by numerical tests in two cases. Not only that, but this method is also very flexible and works equally well with any computer language.

Ahmed et al., (2024) [18] provided a wavelet collocation approach to solving SPDEs and one-parameter SPDEs effectively, considering the control system's intrinsic singular disturbances. It is difficult to analyze and solve these equations because they belong to a family of mathematical models that display both differential and difference equations. By use of Taylor series expansion, the components that include both positive and negative changes were roughly estimated. The accuracy and efficacy of the wavelet collocation approach were shown via numerical tests, highlighting its promise as a dependable tool for studying and solving SPDEs in control systems.

Woldaregay et al., (2024) [19] observed the numerical analysis of SPDEs. The equations under consideration include a minor perturbation parameter $\varepsilon \in (0, 1]$ multiplied by the highest order derivative term, along with shift parameters associated with the non-derivative components. We offer numerical techniques that converge uniformly regardless of the parameter ε. The numerical techniques are developed using the Crank-Nicolson method for temporal discretization and the midpoint upwind non-standard finite difference method on both uniform and Shishkin meshes for spatial discretization. Numerical test cases are used to validate the theoretical results and analysis of the systems.

Lalu et al., (2023) [20] analyzed a method for solving SPDEs using discrete and continuous tiny shift arguments, the solution of which exhibits parabolic boundary layer behavior. The plan was derived by discretizing time using a backward Euler technique and space using a trigonometric spline method. A SPDEs with neighboring SPDEs is obtained by estimating the shift terms using the Taylor series. Comparing numerical results produced using alternative methodologies demonstrates the correctness of the recommended spline numerical methodology.

Ragula et al., (2023) [21] examined a SPDEs of the second order exhibiting both positive and negative changes. The issue is resolved using a fitted non-polynomial spline method. A three-term recurrence relation-based fitted non-polynomial spline method is developed after an approximation version of the issue is obtained using the Taylor series expansion procedure. Researchers record the greatest absolute error as a function of the solution's quadratic rate of convergence.

Ahsan et al., (2023) [22] constructed and enhance a Haar wavelet method of higher order for the solution of nonlinear SPDEs with different sets of boundary conditions, such as initial, boundary, two points, integral, and multipoint integral. On the basis of convergence and accuracy, the suggested higher-order Haar wavelet approach is compared to recently published work, which includes the famous Haar wavelet method. Calculated results are stable, efficient, and of high order, and the suggested approach is simple to apply to other boundary conditions.

Daba et al., (2022) [23] introduced a numerical method for handling SPDEs that exhibit minor mixed shifts that originate from computers science. An oscillation-free higher order numerical scheme is crucial in applications and its development and analysis is the primary goal of this study. A uniformly convergent computational technique is suggested for both spatial and temporal discretization. A comparison with existing approaches in the literature demonstrates the efficacy of the suggested strategy. When compared to existing approaches in literature, the suggested strategy yields more precise findings and a higher order of convergence.

Debela et al., (2020) [24] offered a numerical approach to solving SPDEs. According to the value of the sum of the coefficients in the reaction terms, the solution to this problem displays either layer or oscillatory behavior. Further analysis and graphs illustrating the impact of the delay parameter (small shift) on the boundary layer(s) have also been conducted. By applying it to four model cases, researchers confirm the scheme's applicability. To demonstrate the suggested approach, researchers recorded the maximum absolute errors compared to the other numerical studies.

EXISTENCE OF SOLUTION

In this part, the authors analyze the uniqueness and existence of the solution.

Lemma 1 [25]: The function $u(\tau)$ is the solution to the integral equation if f is a member of $C(\mathcal{T}, R)$, where $\mathcal{T} = [0, T]$.

$$u(\tau) = \phi_1 + \tau\phi_2 + \int_0^\tau (\tau - s)[\lambda_1 f(s) - \lambda_2 u(s)]\, ds, \quad \tau \in \mathcal{T},$$

Assuming that $u(\tau)$ are the responses to problems (2–5) and λ_1, λ_2, and λ_3 are all defined as fractions of $\frac{1}{\epsilon\delta_1}$, $\frac{\delta_2}{\epsilon\delta_1}$, and $\frac{\delta_3}{\epsilon\delta_1}$, respectively.

Proof: The first step in solving the issue given in equation (2-5) is to set it as:

$$u''(\tau) = h(\tau),$$

(6)

Where $h(\tau)$ equal to $\left[\lambda_1 f(s) - \lambda_2 \frac{du(s)}{ds} - \lambda_3 u(s)\right]$ constitute a continuous function. Combining the two sides of (5) from 0 to τ generates

$$u'(\tau) - u'(0) = \int_0^\tau h(s)ds,$$

Or equivalently

$$u'(\tau) = \emptyset_2 + \int_0^\tau h(s)ds.$$

(7)

Calculating the integral of (7) from 0 to τ produces

$$u(\tau) - u(0) = \emptyset_2\tau + \int_0^\tau \int_0^\tau h(s)ds,$$

Or equivalently

$$u(\tau) = \emptyset_1 + \emptyset_2\tau + \int_0^\tau \int_0^\tau (\tau - s)\, h(s)ds,$$

Or equivalently

$$u(\tau) = \emptyset_1 + \emptyset_2\tau + \int_0^\tau (\tau - s)\left[\lambda_1 f(s) - \lambda_2 \frac{du(s)}{ds} - \lambda_3 u(s)\right] ds$$

Researchers examine the following inequality to present the Ulam-Hyers stability:

$$\left|\frac{d^2\bar{u}(\tau)}{d\tau^2} - F\left(\frac{d\bar{u}(\tau)}{d\tau}, \bar{u}(\tau), f(\tau)\right)\right| < \varepsilon$$

(8)

Definitions 1 [26]: A solution to issue (2-5) exhibits "Ulam-Hyers" stability if there occurs a positive real integer μ_k such that for any solution $\bar{u}$ of the inequality (8), there exists an exact solution u satisfying $\mu_k \varepsilon$ is smaller than $\|u - \bar{u}\|$.

To move the issue to a fixed point, they establish the operator $\mathcal{S}: \Omega \to \Omega$ in a way that

$$\mathcal{S}u(\tau) = \emptyset_1 + \emptyset_2\tau + \int_0^\tau (\tau - s)\left[\lambda_1 f(s) - \lambda_2 \frac{du(s)}{ds} - \lambda_3 u(s)\right] ds,$$

In equation (2), the fixed points of the operator $\mathcal{S}$ represent the response. Kindly assume the following to proceed with the analysis. A constant m_1, m_2, m_3, and m_4 exist for any τ that falls inside the interval [0, T] such that

(E1) $\left|\frac{du_1}{d\tau} - \frac{du_2}{d\tau}\right| \leq m_1|u_1 - u_2|$;

(E2) $\left|\frac{du}{d\tau}\right| \leq m_2|u|$;

(E3) $|f(\tau, u_1, u_2) - f(\tau, \bar{u}_1, \bar{u}_2)| \leq m_3|u_1 - \bar{u}_2| + m_4|u_2 - \bar{u}_2|$.

Theorem 2 [27]: Assume that $\mathfrak{T}$ is a compact subset of a Banach space Ω that is nonempty, convex, and compact, and that the compact operator $\mathcal{S}: \mathfrak{T} \to \mathfrak{T}$ transfers $\mathfrak{T}$ into itself. Because of this, $\mathcal{S}$ has a fixed point within $\mathfrak{T}$.

Equation (2) is expressed in its integral form as

$$u(\tau) = \emptyset_1 + \emptyset_2\tau + \int_0^\tau (\tau - s)\left[\lambda_1 f(s) - \lambda_2 \frac{du(s)}{ds} - \lambda_3 u(s)\right] ds, \quad \tau \in \mathcal{T}.$$

The condition $|f(\tau)| \leq e_f$, where e_f is greater than 0, holds because f is a bounded linear function.

1.1 Stability

In this part, the researcher formulates the stability findings for issue (2). Practical problems in physics, economics, and biology all rely on the idea of UH stability. Consider the issue

$$u(\tau) = \emptyset_1 + \emptyset_2\tau + \int_0^\tau (\tau - s)\left[\lambda_1 f(s) - \lambda_2 \frac{du(s)}{ds} - \lambda_2 u(s) + g(s)\right] ds, \quad \tau \in \mathcal{T}$$

(9)

A solution to equation (9) exists for any $\epsilon > 0$ such that $|g(v)| \leq \epsilon$ and g is a member of the set Ω.

$$u(\tau) = \emptyset_1 + \emptyset_2\tau + \int_0^\tau (\tau - s)\left[\lambda_1 f(s) - \lambda_2 \frac{du(s)}{ds} - \lambda_2 u(s) + g(s)\right] ds$$

(10)

By applying the assumptions (E1-E2) to the equation (10), they get

$$u(\tau) = \mathcal{S}u(\tau) + \int_0^\tau (\tau - s)g(s)ds, \ \ \tau \in \mathcal{T}.$$

The inequality $|\mathcal{S}u(\tau) - u(\tau)| \leq \epsilon \frac{\tau^2}{2}$ could be calculated from equation (10), by using the formula (9).

Theorem 3: Assuming that $\frac{\tau^2 \epsilon}{2-\tau^2(|\lambda_2|m_1+\lambda_3||)} < 1$, then problem (2-5) is stable under UH and generalized UH.

Proof: The researchers have an exact solution (u) and an approximation solution $(\bar{u}\epsilon\Omega)$ to the equations (1-4).

$$\|u - \bar{u}\| = \sup_{\tau \in \mathcal{T}} |u(\tau) - \mathcal{S}u(\tau)| \leq + \sup_{\tau \in \mathcal{T}} |u(\tau) - \mathcal{S}u(\tau)| + \sup_{\tau \in \mathcal{T}} |\mathcal{S}u(\tau) - \bar{u}(\tau)| \leq \frac{\tau^2}{2}$$

$$\in \frac{\tau^2}{2}(|\lambda_2|m_1 + |\lambda_3|)\|u - \bar{u}\| \leq \frac{\tau^2 \in}{2 - \tau^2(||\lambda_2|m_1 + |\lambda_3||)}$$

NUMERICAL Method

The researchers study the Laplace transfer method.

The LT Method

Many scholars in the fields of physics and engineering have used LT to examine the solution of differential equations. Here is the definition of the LT of $u(\tau)$ [28]:

$$\hat{u}(s) = \mathrm{L}\{\mathrm{u}(\tau)\} = \int_0^\infty e^{-k\tau}u(\tau)d\tau,$$

Here, ϵ is the LT parameter and the over broad hats are the numbers in Laplace space. The limit of the nth derivative of $u(\tau)$ is provided as.

$$L\left\{\frac{d^n u(\tau)}{d\tau^n}\right\} = s^n\hat{u}(s) - s^{n-1}u(0) - s^{n-2}u'(0), \dots\dots, u^n(0)$$

Applying the LT to equations (2-4), they get

$$L\left\{\in \delta_1 \frac{d^2u(\tau)}{d\tau^2} + \delta_2 \frac{du(\tau)}{d\tau} + \delta_3 u(\tau)\right\} = L\{f(\tau)\}$$

$$\Longrightarrow \in \delta_1[s^2\hat{u}(s) - su(0) - u'(0)] + \delta_2[s\hat{u}(s) - u(0)] + \delta_3\hat{u}(s) = \hat{f}(s)$$

Or else

$$(\in \delta_1 s^2 + \delta_2 s + \delta_3)\hat{u}(s) = \in su(0) + \in u'(0) + u(0) + \hat{f}(s)$$

Simplifying the previous, they get

$$\hat{u}(s) = \frac{\hat{F}(s)}{\in\delta_1 s^2+\delta_2 s+\delta_3}$$

(11)

The equation $\hat{F}(s) =\in su(0)+\in u'(0) + u(0) + \hat{f}(s)$ stands. After plugging Equation 11 into the inverse LT, we get $u(\tau)$ as

$$u(\tau) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} e^{s\tau}\,\hat{u}(s)ds = \frac{1}{2\pi i}\int_{\Gamma} e^{s\tau}\hat{u}(s)ds,$$

(12)

Let Γ be a contour that extends from $\rho - i\infty$ to $\rho > 0$, positioned to the right of all singularities of $\hat{u}(s)$ in the complex s-plane. The main goal is to determine the integral in equation (12), and we also use numerical methods to invert it. A numerical inversion approach must be used in cases if a mathematical inversion of the Laplace domain equation is unavailable. There are several methods described in the literature that can do the Laplace inversion. Each method has its place and is best suited to a certain task. Approximations are used in all inversion procedures to assess the integral provided in Equation (12). The two inversion algorithms that are most often used are highlighted in the below sections.

Euler's Method (EM)

EM is an application of the Fourier series technique that leverages Euler summation for fast convergence. Given that the integral in (12) could be used to derive the Fourier series technique? The approximation $u_{ApSol}(\tau)$ is stated in EM for a given real-valued function $u(\tau)$ and a certain LT $\hat{u}(s)$.

$$u_{ApSol}(\tau) = \frac{10^{\frac{N_e}{3}}}{\tau}\sum_{k=0}^{2N_e}\beta_k R_e\left(\hat{u}\left(\frac{\eta_k}{\tau}\right)\right)$$

(13)

In which

$$\eta_k = \frac{N_e \ln(10)}{3} + \pi i k, \qquad \beta_k = (-1)^k \zeta_k$$

(14)

Through $i = \sqrt{-1}$ and $\zeta_0 = \frac{1}{2}, \zeta_k = 1, 1 \le k \le N_e, \zeta_{2N_e} = \frac{1}{2N_e}$

$$\zeta_{2N_e-k} = \zeta_{2N_e-k+1} + 2^{-N_e}\binom{N_e}{k}, \quad 0 < k < N_e$$

(15)

An analysis of the parameters in Equation (13) and their impact on the precision of the numerical solution was carried out by the authors of [29]. Based on their findings, "for ξ to be essential, ensure that N_e be a positive integer. For the given N_e and system precision, determine the system's accuracy and η_k and β_k in equations (14-5). "In (13) find the value of $u_{ApSol}(\tau)$ for the given function $\hat{u}(s)$ and τ."

Weeks Method (WM)

The WM is one of the easiest and most precise mathematical methods for inversing the LT, dependent upon the selection of two parameters with appropriate values for the Laguerre expansion. In comparison to Talbot's technique and the trapezoidal rule, the WM offers a function expansion—more precisely, the Laguerre series expansion—which is a significant advantage. In the suggested method, researchers get $s = \rho + i\eta$, where η is a real number.

$$u_{ApSol}(\tau) = \frac{e^{\rho\tau}}{2\pi}\int_{-\infty}^{\infty} e^{i\tau\eta}\hat{u}(\rho + i\eta)dy$$

(16)

The authors extend the function $\hat{u}(\rho + \eta)$ as

$$\hat{u}(\rho + \eta) = \sum_{\ell=-\infty}^{\infty} a_\ell \frac{(-\sigma+i\eta)^\ell}{(\sigma+i\eta)^{\ell+1}}, \quad \sigma > 0, \;\; \eta \in \mathbb{R}$$

(17)

By combining (17) with (16), researcher get

$$u_{ApSol}(\tau) = \frac{e^{\rho\tau}}{2\pi}\sum_{\ell=-\infty}^{\infty} a_\ell \delta_\ell(\tau;\sigma)$$

In which

$$\delta_\ell(\tau;\sigma) = \int_{-\infty}^{\infty} e^{\rho\tau}\frac{(-\sigma + i\eta)^\ell}{(\sigma + i\eta)^{\ell+1}}dn$$

When τ is greater than0, researchers can estimate the Fourier integral

$$\delta_\ell(\tau;\sigma) = \begin{cases} 2\pi e^{-\sigma\tau}\mathcal{L}_\ell(2\sigma\tau), & \ell \geq 0, \\ 0, & \ell < 0, \end{cases}$$

In this case, ρ is greater than ρ_0, where ρ_0 is the convergence abscissa and ρ, σ are real numbers and $\mathcal{L}_\ell(\tau)$ is a Laguerre polynomial of degree. It is possible to express the $\mathcal{L}_\ell(\tau)$ as

$$\mathcal{L}_\ell(\tau) = \frac{e^\tau}{\ell!}\frac{d^\tau}{d\tau^\ell}\left(e^{-\tau}\tau^\ell\right),$$

The coefficients for the Taylor series expansion are denoted as a_ℓ.

$$\mathcal{M}(\zeta) = \frac{2\sigma}{1-\zeta}\hat{u}\left(\rho + \frac{2\sigma}{1-\zeta} - \sigma\right) = \sum_{\ell=0}^{\infty} a_\ell \zeta^\ell, \quad |\zeta| < R,$$

(18)

In this case, R stands for the Maclaurin series' radius of convergence (18), and the unknowns a_ℓ are calculated as

$$a_\ell = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\mathcal{M}(\zeta)}{\zeta^{\ell+1}}d\zeta = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{i\beta})e^{-ik\beta}d\beta$$

(19)

The classical Cauchy integral formula, denoted by (19), is numerically calculated as

$$\tilde{a}_\ell = \frac{e^{-i\ell k/2}}{2N_w}\sum_{j=-N_w}^{N_w-1}\mathcal{M}\left(e^{\frac{i\beta_{j+1}}{2}}\right)e^{-i\ell\beta_j},\ \ \ell = 0,1,2,\ldots,N_w-1$$

When β_j is the same as jk and k is the same as the ratio of π to N_w.

NUMERICAL RESULT

Researchers provide here the numerical outcomes of the proposed approaches for SPBVPs that have been computationally evaluated. Using numerical examples, the recommended numerical procedures are shown to be feasible. Our studies were carried out using a "Windows 10 (64-bit) PC equipped with 12.0 GB RAM, Intel (R) Core (TM) i5-3317U 1.70 GHz CPU, and MATLAB R2019a" [30]. To evaluate the precision of the approaches, the maximum absolute error Er is used, defined as Er is equal to $max_{1\le iN}|u(\tau_i) - u_{ApSol}(\tau_i)|$, where $u(\tau)$ represents the analytical solution and $u_{ApSol}(\tau)$ denotes the approximation solution.

Problem 1: In this case, the authors analyze a linear SPPBVP of second order that has the form

$$-\epsilon\frac{d^2u(\tau)}{d\tau^2} + \frac{du(\tau)}{d\tau} = \epsilon\,\pi^2\sin(\pi\tau) + \pi\cos(\pi\tau), \qquad 0 \le \tau \le 1$$

The precise response is $u(\tau) = sin(\pi\tau)$, where $u(0) = 0$ and $u(1) = 0$ are boundary conditions. As seen in Table 1, the numerical approaches that were suggested all produced simulated results.

Table 1: The outcomes of the approaches to Problem 1 simulations.

ε	**10^{-1}**	**10^{-2}**	**10^{-3}**	**10^{-4}**	**10^{-5}**
EM					
$N_e = 15$	7.6888×10^{-11}	7.8918×10^{-11}	8.1356×10^{-11}	8.1217×10^{-11}	7.8536×10^{-11}
$N_e = 16$	2.3088×10^{-11}	2.3306×10^{-11}	2.1312×10^{-11}	2.1526×10^{-11}	2.0640×10^{-11}

$N_e = 17$	4.3501×10^{-12}	4.6905×10^{-12}	4.1645×10^{-12}	1.0237×10^{-12}	3.6222×10^{-12}
$N_e = 18$	6.0057×10^{-13}	4.4890×10^{-12}	4.2960×10^{-12}	1.5062×10^{-12}	4.5998×10^{-12}
WM					
$N_w = 20$	4.4765×10^{-14}	1.1636×10^{-13}	1.2906×10^{-13}	1.2578×10^{-13}	1.8901×10^{-14}
$N_w = 25$	9.3805×10^{-15}	1.8711×10^{-15}	2.5449×10^{-15}	3.3463×10^{-15}	2.0640×10^{-15}
$N_w = 30$	5.8976×10^{-16}	8.5283×10^{-16}	2.8222×10^{-16}	5.7803×10^{-16}	1.0008×10^{-17}
$N_w = 35$	2.7933×10^{-16}	3.8885×10^{-16}	6.2709×10^{-16}	8.4146×10^{-16}	4.2057×10^{-17}
[31]	8.72×10^{-7}	3.41×10^{-7}	1.42×10^{-6}	1.68×10^{-6}	1.72×10^{-6}

Figures 2 and 3 provide a comparison of Er acquired using the EM and the WM for different values of ϵ. They see that the accuracy improves with increasing node counts for all proposed numerical approaches. All approaches consistently provide reliable outcomes. Figure 5 displays the Er of the numerical schemes that were proposed concerning the Er of the finite difference scheme [31]. Consistent results are obtained by using the numerical approaches suggested.

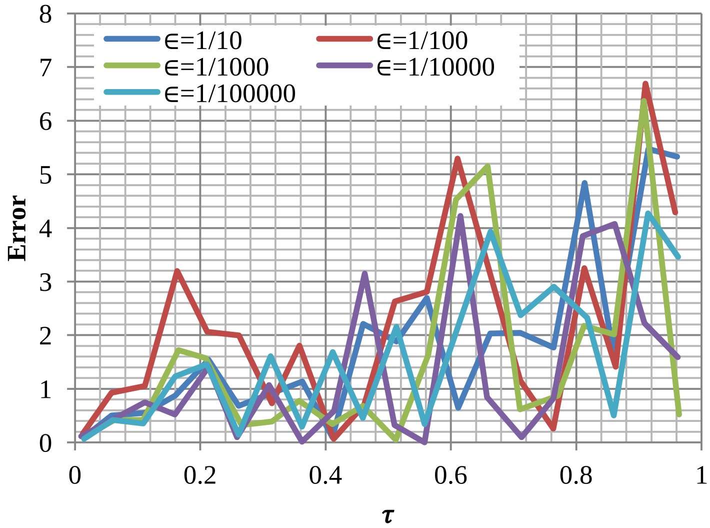

Figure 2: Using $N_e = 18$, the absolute error plotted using WM for Problem 1

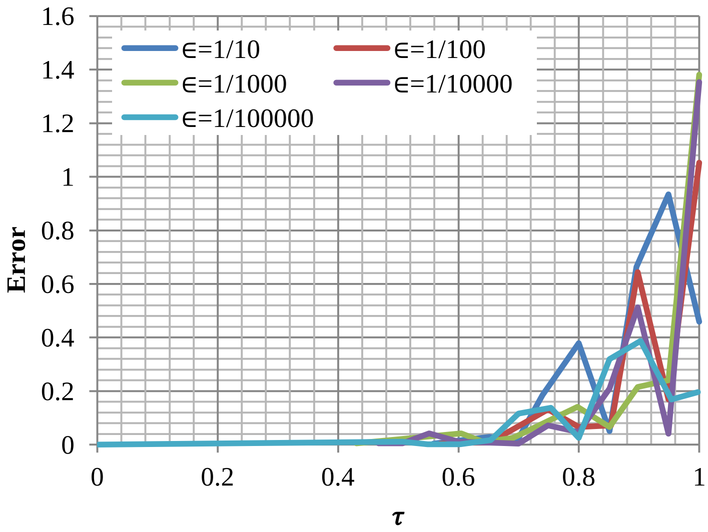

Figure 3: Using $N_w = 20$, the absolute error plotted using the WM for Problem 1.

Problem 2: In this case, researchers assess a linear SPBVP of the second order that has the form

$$\frac{d^2u(\tau)}{d\tau^2} + \in u(\tau) = 0, \quad 0 \leq \tau \leq 1,$$

Given that $u(0) = 1$ and $u(1) = 1$, the precise response is $u(\tau)$ is equal to the ratio of $sin\frac{\tau}{\sqrt{\epsilon}}$ to $sin\frac{1}{\sqrt{\epsilon}}$. As shown in Table 2, the suggested numerical approaches' simulation results are provided.

Table 2: The outcomes of the approaches to Problem 2 simulations

ε	**2^{-2}**	**2^{-3}**	**2^{-4}**	**2^{-5}**	**2^{-6}**
EM					
$N_e = 14$	4.6103×10^{-11}	2.0156×10^{-9}	1.8342×10^{-10}	3.7054×10^{-10}	8.6851×10^{-10}
$N_e = 16$	7.5725×10^{-12}	1.1186×10^{-11}	2.1761×10^{-11}	4.4601×10^{-11}	7.1315×10^{-11}
$N_e = 18$	1.8157×10^{-12}	6.8196×10^{-12}	4.3202×10^{-12}	4.3526×10^{-12}	1.5897×10^{-12}
$N_t = 60$	2.3300×10^{-13}	5.3272×10^{-13}	5.1985×10^{-13}	7.5566×10^{-13}	1.3742×10^{-12}
WM					
$N_W = 25$	3.3307×10^{-16}	2.8866×10^{-16}	6.6280×10^{-14}	5.1025×10^{-15}	6.3298×10^{-9}
$N_W = 30$	4.3409×10^{-16}	0	4.1299×10^{-15}	1.5987×10^{-15}	3.3111×10^{-9}

$N_w = 35$	0	2.2204×10^{-16}	6.5613×10^{-15}	2.6654×10^{-15}	1.1020×10^{-14}
[32]	5.9287×10^{-7}	3.7321×10^{-6}	2.3293×10^{-6}	5.7657×10^{-5}	9.5596×10^{-5}

Figures 4 and 5 provide the graphical representation of Er acquired using the EM and the WM for different values of ϵ. Comparing the Er of the suggested numerical techniques with the Er of GRBFM [32] for various parameter values is shown in Table 4. The EM and WM have done a better job than GRBFM at approximating the solution of SPBVPs [32].

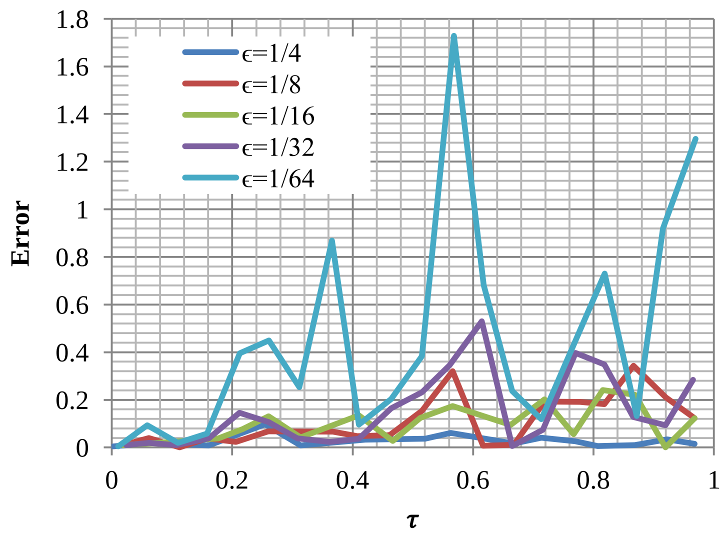

Figure 4: Using N_e equal to 22, absolute error as a function of EM for Problem 2

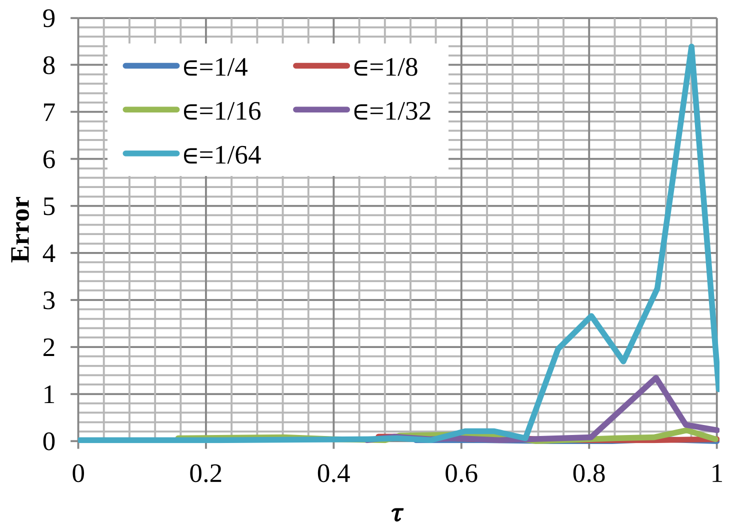

Figure 5: The absolute error plot using the WM for Problem 2 via $N_w = 35$

CONCLUSION

The study successfully demonstrates the effectiveness of LTM, particularly EM and WM, for solving SPBVPs. These problems pose significant numerical challenges due to the presence of thin boundary layers, where traditional numerical methods often fail to achieve uniform accuracy. The proposed methods effectively handle these difficulties by leveraging the LT and its numerical inversion techniques. The results obtained from numerical experiments confirm the stability and high accuracy of EM and WM across varying values of the perturbation parameter. A comparative analysis with existing numerical schemes further validates the superior performance of the proposed methods, showing reduced absolute errors and improved convergence rates. The findings indicate that the suggested techniques provide a reliable framework for addressing SPBVPs, making them highly applicable in fields such as fluid dynamics, quantum mechanics, and control systems. Future work may explore extending these approaches to more complex singular perturbation problems, incorporating adaptive numerical techniques for enhanced efficiency and robustness.

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