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The Spectrum of a Matrix Differential Operator

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Abstract

This paper extends the spectral theory of differential operators to encompass matrix differential operators derived from formally self-adjoint matrix differential expressions. Building on the seminal work of Choudhary and Everitt [1], the study establishes conditions under which the spectrum of such operators can be rigorously characterized. By constructing the Green's matrix and employing variational principles, we derive detailed eigenvalue bounds and investigate the influence of boundary conditions and domain variations on the spectral properties. In particular, we show that the associated differential operator is symmetric and self-adjoint in an appropriate Hilbert space and that its spectrum is discrete under natural conditions. [1, 3, 7].

Keywords: Matrix Differential Operator; Spectral Theory; Self-Adjoint Operator; Green's Matrix; Boundary Value Problem; Discrete Spectrum; Variational Methods; Eigenvalue Estimation; Meromorphic Functions; Limit-Point Case.

1. Introduction

The spectral theory of differential operators has been central to both analysis and mathematical physics. In the scalar setting, classical results guarantee that self-adjoint operators under suitable boundary conditions have a real and discrete spectrum [5, 6]. For instance, Choudhary and Everitt [1] studied fourth-order self-adjoint differential operators under conditions including the real-valuedness and absolute continuity of coefficient functions.

$$N(y) \equiv y^{(4)}(x) - \left[p_1(x) \cdot y^{(1)}(x)\right]^{(1)} + q_1(x) \cdot y(x), 0 \le x < \infty \qquad (1.1)$$

where

$$y^r(x) \equiv \frac{d^r(y)}{dx^r}, \qquad (r = 1,2,3,4), \qquad (1.2)$$

with the following conditions:
(i) $p_1(x), q_1(x)$ are real-valued functions of x defined on $[0, \infty)$,
(ii) $p_1(x)$ is absolutely continuous on any compact subinterval of $[0, \infty)$ (so that $p_1^{(1)}(x)$ exists almost everywhere on$[0, \infty)$ and $q_1(x)$ belongs to $L[0, X)$ for all $X > 0$,
(iii) $q_1(x)$ is bounded below, say $q_1(x) \ge \alpha$ for all x in $[0, \infty)$,

(iv) $0 \leq p_1(x) \leq Kx^2|q_1(x)|^{\frac{1}{2}}$ for all sufficiently large x and for some $K > 0$,
(v) $q_1(x)$ steadily increases and tends to ∞ as x tends to ∞.

2. We consider the matrix differential expression

$$M\phi(x), 0 \leq x < \infty \tag{2.1}$$

where M stands for the matrix differential operator given by

$$M = \begin{bmatrix} -\frac{d}{dx}\left(p_0 \frac{d}{dx}\right) + p_1(x) & r(x) \\ r(x) & -\frac{d}{dx}\left(q_0 \frac{d}{dx}\right) + q_1(x) \end{bmatrix} \tag{2.2}$$

and $\phi(x)$ a vector represented by a column matrix $\begin{bmatrix} u \\ v \end{bmatrix}$.
We assume the following conditions to be satisfied:

(i) $p_0(x), q_0(x), p_1(x), q_1(x), r(x)$ are real valued functions of x defined on $[0, \infty)$,

(ii) $p_0(x)$ and $q_0(x)$ are absolutely continuous on $[0, b]$ for all $b > 0$,

(iii) $p_1(x)$, $q_1(x)$ belong to $L[0, b]$ for all $b > 0$,
(iv) $p_1(x), q_1(x), r(x)$ are bounded below say $p_1(x), q_1(x), r(x) \geq \alpha > 0$, for all $x \in [0, \infty)$,
(v) $r(x) \leq p_1(x), q_1(x)$ for all $x \in [0, \infty)$,
(vi) $0 < p_0(x), q_0(x) < Kx^2$ for all sufficiently large x and for some $K > 0$,
(vii) $p_1(x)$ and $q_1(x)$ increases steadily and tend to ∞ as $x \to \infty$.

If $p_0(x), q_0(x), p_1(x), q_1(x), r(x)$ all satisfy the conditions (i) – (vi), then Shaw and Bhagat [2] have proved that the matrix differential equation

$$M\phi(x) = \lambda\phi(x), 0 \leq x < \infty \tag{2.3}$$

has exactly two linearly independent solutions for each x such that $im\,(\lambda) \neq 0$, which belong to the class $L^2[0, \infty)$, that is, M is in the $limit - 2$ case at infinity.

Let the differential operator T associated with M be defined as follows:

$$Tf = Mf, \forall f \in D(T) \tag{2.4}$$

where $D(T)$ denote the domain of T, which is the set of all vector functions

$$f \equiv f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$$

satisfying the following:

(a) $f(x) \in L^2[0, \infty)$,
(b) $f'(x)$ is absolutely continuous on $[0, b]$ for all $b > 0$,
(c) $Mf \in L^2[0, \infty)$,
(d) $f(0) = 0$ or $f'(0) = 0$.

Then T is a symmetric and self-adjoint operator in the Hilbert space $L^2[0,\infty)$ of vector. The eigenvalues of the operator T, defined by (2.3), are those points μ, where at least one of the $k_{rs}(\mu)$ has a jump-discontinuity. This is the case if the corresponding $m_{rs}(\mu)$ has a pole.

3. The Green's matrix $G(x,y,\lambda)$ for the boundary value problem, defined by

$$(M-\lambda)\phi(x)=0 \tag{3.1}$$

$f(0)=0$ or $f'(0)=0, \phi(x)\in L^2[0,\infty)$ is given by

$$G(x,y,\lambda)=\begin{bmatrix}G_{11} & G_{21}\\ G_{12} & G22\end{bmatrix}=\begin{bmatrix}\psi_{11}(x,\lambda) & \psi_{21}(x,\lambda)\\ \psi_{12}(x,\lambda) & \psi_{22}(x,\lambda)\end{bmatrix}\cdot\begin{bmatrix}u_1(y,\lambda) & v_1(y,\lambda)\\ u_2(y,\lambda) & v_2(y,\lambda)\end{bmatrix};y\in[0,\infty)$$

$$=\begin{bmatrix}u_1(x,\lambda) & u_2(x,\lambda)\\ v_1(x,\lambda) & v_2(x,\lambda)\end{bmatrix}\cdot\begin{bmatrix}\psi_{11}(y,\lambda) & \psi_{12}(y,\lambda)\\ \psi_{21}(y,\lambda) & \psi_{22}(y,\lambda)\end{bmatrix};y\in[0,\infty) \tag{3.2}$$

Since $\phi_j(x,\lambda),\theta_j(x,\lambda)$ $(j=1,2)$ are analytic functions of λ, real for real λ, they are integral functions of λ. Therefore, the Green's matrix $G(x,y,\lambda)$ is a meromorphic functions of λ if and only if $m_{rs}(\lambda)$ $(1\le r,s\le 2)$ are meromorphic functions of λ.

4. To develop a theory for the singular boundary value problem (3.1) we analyse the boundary value problem defined on a finite interval $[0,b]$ and then let $b\to\infty$. Consider the boundary value problem on $[0,b]$,

$$\left.\begin{matrix}(M-\lambda)\phi(x)=0\\ \phi(0)=0,\qquad \phi(b)=0\end{matrix}\right\} \tag{4.1}$$

Let $\phi=\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$ and $\theta=\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$ be two vectors having continuous derivatives of second order and A^T denote the transpose of the matrix A. Then the Green's formula for our boundary value problem is

$$\int_a^b(\theta^T M\phi-\phi^T M\theta)\,dx=[\phi\theta](0) \tag{4.2}$$

Let $\phi_r(x)=\begin{bmatrix}f_r(x)\\ g_r(x)\end{bmatrix}$, $(r=1,2)$ belong to $L^2[0,b]$ and such that $\phi_r(x),(r=1,2)$ are absolutely continuous and $M\phi_r(x),(r=1,2)$ are $L^2[0,b]$. We suppose that

$$\alpha_1 = \begin{bmatrix} p_0^{\frac{1}{2}} & f_1' \\ q_0^{\frac{1}{2}} & g_1' \end{bmatrix}, \alpha_2 = \begin{bmatrix} p_0^{\frac{1}{2}} & f_2' \\ q_0^{\frac{1}{2}} & g_2' \end{bmatrix}, \beta_1 = \begin{bmatrix} p^{\frac{1}{2}} & f_1 \\ q^{\frac{1}{2}} & g_1 \end{bmatrix}, \beta_2 = \begin{bmatrix} p^{\frac{1}{2}} & f_2 \\ q^{\frac{1}{2}} & g_2 \end{bmatrix},$$

$$\gamma_1 = \begin{bmatrix} r^{\frac{1}{2}} & f_1 \\ r^{\frac{1}{2}} & g_1 \end{bmatrix}, \gamma_2 = \begin{bmatrix} r^{\frac{1}{2}} & f_2 \\ r^{\frac{1}{2}} & g_2 \end{bmatrix}, \beta_1^* = \begin{bmatrix} p_0^{\frac{1}{2}} & f_1 \\ q_0^{\frac{1}{2}} & g_1 \end{bmatrix}, \beta_2^* = \begin{bmatrix} p_0^{\frac{1}{2}} & f_2 \\ q_0^{\frac{1}{2}} & g_2 \end{bmatrix},$$

$$\gamma_1^* = \begin{bmatrix} r^{\frac{1}{2}} & g_1 \\ r^{\frac{1}{2}} & f_1 \end{bmatrix}, \gamma_2^* = \begin{bmatrix} r^{\frac{1}{2}} & g_2 \\ r^{\frac{1}{2}} & f_2 \end{bmatrix}$$

and define

$$D_b(\phi_1, \phi_2) = \int_0^b \{\alpha_1^T \alpha_2 + \beta_1^T \beta_2 + \gamma_1^T \gamma_2^*\}\, dx \tag{4.3}$$

$$D_b(\phi_1, \phi_1) = \int_0^b \{\alpha_1^T \alpha_1 + \beta_1^T \beta_1 + \gamma_1^T \gamma_1^*\}\, dx \tag{4.4}$$

We arrange the eigenvalues in non-decreasing order and suppose that the eigenvalues and the corresponding normalized eigenvectors of the boundary value problem (4.1) be $\{\lambda_{n,b}, n \geq 1\}$ and $\{\psi_n(b,x); n \geq 1\}$ respectively. If $N_b(\lambda)$ denote the number of eigenvalues $\lambda_{n,b}$ not exceeding λ, then our aim is to prove that $N_b(\lambda)$ remains bounded for every given λ and for all sufficiently large b.

5. In the present section, we prove some properties of $D_b(\phi) \equiv D_b(\phi, \phi)$.

Lemma 5.1 (i) If $\Psi_n = \Psi_n(b; x)$, then $D_b(\Psi_m, \Psi_n) = 0$ if $m \neq n$.

Proof: We can easily verify, on integration by parts that

$$\int_0^b \{\alpha_1{}^T \alpha_2 + \beta_1{}^T \beta_2 + \gamma_1{}^T \gamma_2{}^*\}\, dx = [\alpha_1{}^T \beta_2{}^*]_0^b + \int_0^b \varphi_2{}^T M\, \varphi_1 dx \tag{5.1.1}$$

The integrated term on the right-hand side of (5.1.1) vanishes due to the boundary conditions of (4.1), [Thus we have, by replacing φ_1 and φ_2 by ψ_m and ψ_n respectively]

$$D_b(\psi_m, \psi_n) = \int_0^b \psi_m{}^T M\, \psi_m\, dx = \lambda_{m,b} \int_0^b \psi_n{}^T \psi_m\, dx = 0$$

Hence the result.

(ii) $D_b(\psi_n) = \lambda_{n,b}$.

To prove (ii), we note from above that when $m = n$.

$$D_b(\psi_n, \psi_n) = \lambda_{n,b} \int_0^b \psi_n{}^T \psi_n\, dx = \lambda_{n,b},$$

since the eigenvectors are normalized.

Hence the result.

(iii) $D_b(\alpha\psi_m + \beta\psi_n) = \alpha^2\lambda_{m,b} + \beta^2\lambda_{n,b}$ if $m \neq n$
$= (\alpha+\beta)^2\lambda_{n,b}$ if $m = n$.

Proof: $D_b(\alpha\psi_m + \beta\psi_n) = \alpha^2 D_b(\psi_m) + \beta^2 D_b(\psi_n) + 2\alpha\beta D_b(\psi_m, \psi_n)$.
If $m \neq n$, then using (i) and (ii) we get

$$D_b(\alpha\psi_m + \beta\psi_n) = (\alpha+\beta)^2\lambda_{n,b}$$

Hence the result.

(iv) $D_b(\psi_n)$ is bounded below for all n if $p_0(x), q_0(x), p_1(x), q_1(x), r(x)$ is bounded below.

Proof: $D_b(\psi_n) = \int_0^b \left[p_0\left(\psi_{n1}{}'\right)^2 + q_0\left(\psi_{n2}{}'\right)^2 + p_1(\psi_{n1})^2 + q_1(\psi_{n2})^2 + 2r\psi_{n1}\psi_{n2}\right] dx$
If $p_0(x), q_0(x), p_1(x), q_1(x), r(x) \geq a$

$$D_b(\psi_n) \geq a\int_0^b \left[\left(\psi_{n1}{}'\right)^2 + \left(\psi_{n2}{}'\right)^2 + (\psi_{n1})^2 + (\psi_{n2})^2 + 2\psi_{n1}\psi_{n2}\right] dx$$

$$= a\int_0^b \left[\psi_n'^{T}\psi_n{}' + \psi_n{}^T\psi_n + 2\psi_{n1}\psi_{n2}\right] dx$$

As ψ_n and $\psi_n{}'$ are $L^2 0,\infty)$. Hence it is bounded. Therefore, we conclude that eigenvalues are bounded below.

Lemma 5.2.

Let $D(b)$ be the set of all vector functions $\phi(x)$ such that
(i) $\phi(x) \in L^2[0,b]$,
(ii) $\phi'(x)$ is absolutely continuous on $[0,b]$,
(iii) $\phi(0) = \phi(b) = 0$.

If $c_{n,b} = \int_0^b \psi_n{}^T\phi(x)\,dx$ be the Fourier coefficient of $\phi(x)$ corresponding to the boundary value problem (4.1), then for all $\phi(x)$ belonging to $D(b)$,
(a) $D_b(\phi, \psi_n) = \lambda_{n,b}c_{n,b}$,
(b) If $p_0(x), q_0(x), p_1(x), q_1(x), r(x) \geq a > 0$ then $D_b(\phi) \geq \lambda_{n,b}c_{n,b}^2$.

Proof: We have

$$D_b(\phi, \psi_n) = \lambda_{n,b}\int_0^b \phi^T\,\psi_n\,dx = \lambda_{n,b}c_{n,b}.$$

Hence the result (a) follows.

We now proceed to prove the result (b). Firstly

$$D_b\left(\phi-\sum_{n=1}^{m}c_{n,b}\psi_n\right)$$
$$=\int_0^b\left[p_0\left(f_1{}'-\sum_{n=1}^{m}c_{n,b}\psi_{n1}{}'\right)^2+q_0\left(g_1{}'-\sum_{n=1}^{m}c_{n,b}\psi_{n2}{}'\right)^2\right.$$
$$+p\left(f_1-\sum_{n=1}^{m}c_{n,b}\psi_{n1}\right)^2+q\left(g_1-\sum_{n=1}^{m}c_{n,b}\psi_{n2}\right)^2$$
$$\left.+2r\left(f_1-\sum_{n=1}^{m}c_{n,b}\psi_{n1}\right)\left(g_1-\sum_{n=1}^{m}c_{n,b}\psi_{n2}\right)\right]dx$$

Since $p_0(x),q_0(x),p_1(x),q_1(x),r(x)\geq a>0$

$$D_b\left(\phi-\sum_{n=1}^{m}c_{n,b}\psi_n\right)$$
$$\geq a\int_0^b\left[\left(f_1{}'-\sum_{n=1}^{m}c_{n,b}\psi_{n1}{}'\right)^2+\left(g_1{}'-\sum_{n=1}^{m}c_{n,b}\psi_{n2}{}'\right)^2\right.$$
$$+\left(f_1-\sum_{n=1}^{m}c_{n,b}\psi_{n1}\right)^2+\left(g_1-\sum_{n=1}^{m}c_{n,b}\psi_{n2}\right)^2$$
$$\left.+2\left(f_1-\sum_{n=1}^{m}c_{n,b}\psi_{n1}\right)\left(g_1-\sum_{n=1}^{m}c_{n,b}\psi_{n2}\right)\right]dx$$

i.e. $D_b\left(\phi-\sum_{n=1}^{m}c_{n,b}\psi_n\right)\geq a\int_0^b[\text{a positive quantity}]\ dx$

Hence

$$D_b\left(\phi-\sum_{n=1}^{m}c_{n,b}\psi_n\right)\geq 0$$

Now

$$D_b\left(\phi-\sum_{n=1}^{m}c_{n,b}\psi_n\right)=D_b(\phi)+\sum_{n=1}^{m}\lambda_{n,b}c_{n,b}^2-2\sum_{n=1}^{m}\lambda_{n,b}c_{n,b}^2$$
$$=D_b(\phi)-\sum_{n=1}^{m}\lambda_{n,b}c_{n,b}^2$$

Therefore,

$$D_b(\phi)\geq\sum_{n=1}^{\infty}\lambda_{n,b}c_{n,b}^2.$$

Lemma 5.3.

Let $D_M(b)$ be the set of all vector functions $\phi(x)=\begin{bmatrix}f_1(x)\\g_1(x)\end{bmatrix}$ satisfying

(i) $\phi(x)$ belongs to $L^2 0,\infty)$.

(ii) $\phi(x)$ is absolutely continuous in $[0, b]$.
(iii) $M\phi$ belongs to $L^2 0, \infty)$.
(iv) $\phi(0) = \phi(b) = 0$.

Then for all $\phi(x)$ belonging to $D_M(b)$, $D_b(\phi) = \sum_{n=1}^{\infty} \lambda_{n,b} c_{n,b}^2$.

Proof: By Green's formula (4.2), we have

$$\int_0^b [\psi_n^T M\phi - \phi^T M\psi_n]\, dx = [\psi_n \phi](b) - [\psi_n \phi](a) = 0,$$

by virtue of condition (iv).

Hence

$$\int_0^b \psi_n^T M\phi\, dx = \int_0^b \phi^T M\psi_n\, dx \tag{5.3.1}$$

Let

$$c_{n,b}^* = \int_0^b \psi_n^T M\phi\, dx,$$

then

$$c_{n,b}^* = \int_0^b \phi^T M\psi_n\, dx, \qquad \text{by (5.2.1)}$$

$$= \lambda_{n,b} \int_0^b \phi^T \psi_n\, dx = \lambda_{n,b} c_{n,b}.$$

$$\int_0^b \phi^T M\phi\, dx = \frac{1}{4}\int_0^b [(\phi + M\phi)^T(\phi + M\phi) - (\phi - M\phi)^T(\phi - M\phi)]\, dx$$

$$= \frac{1}{4}\left[\sum_{n=1}^{\infty} (c_{n,b}^* + c_{n,b})^2 - \sum_{n=1}^{\infty} (c_{n,b} - c_{n,b}^*)^2\right] \tag{5.3.2}$$

(by Perceval's formula)

$$\int_0^b \phi^T M\phi\, dx = D_b(\phi) - [\alpha_1^T \beta_1^*]_0^b = D_b(\phi) \tag{5.3.3}$$

Since the second term vanishes due to the condition (iv). Therefore, from (5.3.2) and (5.3.3), we have $D_b(\phi) = \sum_{n=1}^{\infty} \lambda_{n,b} c_{n,b}^2$.

6. We now consider the change in the eigenvalues with the change in the boundary value problem (4.1) by increasing $p(x), q(x)$ and $r(x)$ such that $r(x) \le p_1(x) q_1(x)$. Let $\lambda_{n,b}, \psi_n(b; x), c_{n,b}$ be the nth eigenvalue, the nth normalized eigenvector and the nth Fourier coefficient of $\phi(x)$ respectively corresponding to the boundary value problem (4.1). Similarly, let $\mu_{n,b}, \chi_n(b; x)$ be the nth eigenvalue and nth normalized eigenvector of the boundary value problem (4.1) with $p(x), q(x)$ and $r(x)$ replaced by $P(x), Q(x)$ and $R(x)$ respectively such that $p_1(x) \le P(x), q_1(x) \le Q(x)$, $r(x) \le R(x), R(x) \le P(x), Q(x)$. We shall prove below a lemma in which $p_1(x), q_1(x)$ and $r(x)$ will be increased. Similar proof can be obtained by increasing $p_1(x), q_1(x)$ and $r(x)$ alone.

Lemma 6.1: The nth eigenvalue increases as $p(x), q(x)$ and $r(x)$ increases.

Proof: Let $d_{n,b}$ be the Fourier coefficient of $\phi(x)$ corresponding to the boundary value problem obtained from (4.1) by replacing $p_1(x), q_1(x)$ and $r(x)$ by $P(x), Q(x)$ and $R(x)$ respectively. Writing $D_b(\phi)$ in terms of the coefficients $p_1(x), q_1(x)$ etc. We have

$$D_b(\phi; p, q, r) = \int_0^b (\alpha_1^T \alpha_1 + \beta_1^T \beta_1 + \gamma_1^T \gamma_1^*)\ dx$$

$$= \int_0^b \left(p_0 f'^2 + q_0 g'^2 + p_1 f^2 + q_1 g^2 + 2rfg\right) dx \qquad (6.1.1)$$

$$D_b(\phi, P, Q, R) = \int_0^b \left(p_0 f'^2 + q_0 g'^2 + P f^2 + Q g^2 + 2Rfg\right) dx \qquad (6.1.2)$$

But $p_1(x), q_1(x)$ and $r(x)$ tend to ∞ as x tends to ∞, therefore, for large x,

$$D_b(\phi, p_1, q_1, r) \leq D_b(\phi, P, Q, R) \qquad (6.1.3)$$

Let

$$\phi(x) = \begin{bmatrix} \chi_{11}(x) \\ \chi_{12}(x) \end{bmatrix} = \chi_1(x),$$

then

$$\lambda_{1,b} = \lambda_{1,b} \int_0^b \chi_1^T \chi_1 dx = \lambda_{1,b} \sum_{n=1}^{\infty} c_{n,b}^2 \leq \sum_{n=1}^{\infty} \lambda_{n,b} c_{n,b}^2 \leq D_b(\phi, p_1, q_1, r)$$

by Lemma 5.2(b). From (6.1.3), we have

$$\lambda_{1,b} \leq D_b(\phi, P, Q, R) = \mu_{1,b}.$$

Thus

$$\lambda_{1,b} \leq \mu_{1,b}.$$

Next suppose that

$$\phi(x) = d_{1,b}\chi_1 + d_{2,b}\chi_2,$$

where $d_{1,b}$ and $d_{2,b}$ are constants such that

$$d_{1,b}^2 + d_{2,b}^2 = 1 \qquad (6.1.4)$$

Then

$$c_{1,b} = d_{1,b}A + d_{2,b}B \text{ where } A = \int_0^b \chi_1^T \psi_1\, dx,\ B = \int_0^b \chi_2^T \psi_1\, dx,$$

We can choose $d_{1,b}$ and $d_{2,b}$ such that $c_{1,b} = 0$. If A and B are not both zero, then solution of (6.1.4) is given by

$$d_{1,b} = \frac{B}{\sqrt{A^2 + B^2}}, d_{2,b} = -\frac{A}{\sqrt{A^2 + B^2}} \qquad (6.1.5)$$

If A and B are both zero, then $c_{1,b} = 0$ for all $d_{2,b}$ satisfying (6.1.4), we now have

$$\sum_{n=1}^{\infty} c_{n,b}^2 = \int_0^b \left(d_{1,b}\chi_1 + d_{2,b}\chi_2\right)^T \left(d_{1,b}\chi_1 + d_{2,b}\chi_2\right) dx$$

$$= \int_0^b \left(d_{1,b}^2 \chi_1^T \chi_1 + d_{2,b}^2 \chi_2^T \chi_2\right)\ dx$$

$$= d_{1,b}^2 + d_{2,b}^2 = 1, \qquad \text{by } (6.1.4)$$

Therefore,

$$\sum_{n=1}^{\infty} \lambda_{n,b} c_{n,b}^2 = \sum_{n=2}^{\infty} \lambda_{n,b} c_{n,b}^2 \geq \lambda_{2,b} \sum_{n=2}^{\infty} c_{n,b}^2 = \lambda_{2,b}.$$

Hence, using the result (b) of the Lemma 5.2, we have

$$\lambda_{2,b} \leq \sum_{n=2}^{\infty} \lambda_{n,b} c_{n,b}^2 \leq D_b(\phi, p_1, q_1, r) \leq D_b(\phi, P, Q, R) = \mu_{1,b} d_{1,b}^2 + \mu_{2,b} d_{2,b}^2,$$

(by Lemma 5.1(iii))

i.e. $\lambda_{2,b} \leq \mu_{2,b}\left(d_{1,b}^2 + d_{2,b}^2\right) = \mu_{2,b}.$

Therefore, $\lambda_{2,b} \leq \mu_{2,b}.$

To prove, in general $\lambda_{2,b} \leq \mu_{2,b}$, we have to suppose that

$$\phi(x) = d_{1,b}\chi_1 + d_{2,b}\chi_2 + \cdots + d_{n,b}\chi_n$$

where $d_{1,b}, d_{2,b}, \cdots, d_{n,b}$ are constraints such that

$$d_{1,b}^2 + d_{2,b}^2 + \cdots + d_{n,b}^2 = 1 \tag{6.1.6}$$

And such that $(n-1)$ conditions of the form

$$\left.\begin{aligned} A_1 d_{1,b} + \cdots + M_1 d_{n-1,b} + N_1 d_{n,b} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{n-1} d_{1,b} + \cdots + M_{n-1} d_{n-1,b} + N_{n-1} d_{n,b} = 0 \end{aligned}\right\} \tag{6.1.7}$$

are satisfied. Suppose that the determinant $|A_1, \cdots, M_{n-1}|, |B_1, \cdots, N_{n-1}|, \cdots$ do not all vanish. If, for example, the first does not all vanish, and then the equations

$$A_1 \frac{d_{1,b}}{d_{n,b}} + - - - - + M_1 \frac{d_{n-1,b}}{d_{n,b}} = -N_1 etc.$$

can be solved for $\frac{d_{1,b}}{d_{n,b}}, \cdots$ and so (6.1.6) and (6.1.7) can be satisfied. On the other hand, if $|A_1, \cdots, M_{n-1}| = 0$. Let $d_{n,b} = 0$. Then the system (6.1.7) has a non-null solution, which can be normalized to satisfy (6.1.6). Thus $\lambda_{n,b} \leq \mu_{n,b}$ for all n.

Therefore, each eigenvalue increases as $p_1(x), q_1(x)$ and $r(x)$ increases. In the same way it can be proved that each eigenvalue increases as $p_1(x)$ or $q_1(x)$ or $r(x)$ alone increases.

7: Variation of the eigenvalue with the interval: We now observe the change in the eigenvalue as the fundamental interval $[0, b]$ increases.

Lemma 7.1: The *nth* eigenvalue decreases as the fundamental interval increases.

Proof: Let $\lambda_{n,b'}$, $\psi_n(b'; x)$, $c_{n,b'}$ be respectively the *nth* eigenvalue, the *nth* normalized eigenvector and the *nth* Fourier coefficient of $\phi(x)$ for the boundary value problem obtained from (3.1) by respectively b by b'.

Now suppose that

$$\begin{aligned} \phi(x) &= \psi_1(b; x), \quad 0 \leq x \leq b' \\ &= 0, \quad b \leq x \leq b'. \end{aligned}$$

Therefore $\phi(x)$ belongs to $D(b')$. Therefore, by the *(b)* of Lemma 1.5.2, we have

$$D_{b'}(\phi) \geq \sum_{n=1}^{\infty} \lambda_{n,b'} c_{n,b'}^2$$

Also, we have

$$\lambda_{1,b} = D_b(\psi_1) = D_b(\phi) = D_{b'}(\phi) \geq \sum_{n=1}^{\infty} \lambda_{nb'} c_{n,b'}^2 \geq \lambda_{1,b'} \sum_{n=1}^{\infty} c_{n,b'}^2 = \lambda_{1,b'} \int_0^{b'} \phi^T \phi \, dx$$

$$= \lambda_{1,b'} \left[\int_0^b \phi^T \phi \, dx + \int_b^{b'} \phi^T \phi \, dx \right] = \lambda_{1,b'} \int_0^b \psi_1^T \psi_1 \, dx = \lambda_{1,b'}$$

Thus, we see that if $b \leq b'$, then $\lambda_{1,b} \geq \lambda_{1,b'}$.

Then general result can now be proved in the same way as in Lemma 6.1.

8: In this section, we obtain the lower bound of the eigenvalue $\lambda_{n,b}$. To obtain it we divide the fundamental interval $[a, b]$ into subintervals $[x_{r-1}, x_r]$ $(r = 1,2, \cdots, m)$, where $x_1 = a, x_m = b$ and from the boundary value problem on each of these subintervals as follows:

$$\left.\begin{aligned} M\phi(x) &= \lambda\phi(x) \\ \phi'(x_r) = \phi'(x_{r-1}) &= 0 \end{aligned}\right\} \tag{8.1}$$

Let μ_n^r, χ_n^r, d_n^r be respectively the nth eigenvalue, nth normalized eigenvectors and the nth Fourier coefficients of $\phi(x)$ corresponding to the boundary value problem (1.8.1). We also arrange the numbers μ_n^r $(r = 1,2, \cdots, m; n \geq 1)$ in non-decreasing order.

Lemma 8.1: if $\mu_n{}'$ denote the nth member of the non-decreasing sequence formed by the numbers $\{\mu_n^r, r = 1,2, \cdots, m; n \geq 1\}$, then $\lambda_{n,b} \geq \mu_n'$.

Proof: Let us write

$$D^r(\phi) \equiv D^r(\phi, \phi) = \int_{x_{r-1}}^{x_r} [\alpha_1^T \alpha_1 + \beta_1^T \beta_1 + \gamma_1^T \gamma_1^*] \, dx \tag{8.1.1}$$

Obviously,

$$D_b(\phi) = \sum_{s=1}^{m} D^r(\phi) \tag{8.1.2}$$

First of all, we prove that

$$\lambda_{1,b} \geq \mu_1' \tag{8.1.3}$$

To prove the result (8.1.3), let $\phi(x) = \psi_1(b; x)$ for $0 \leq x \leq b$. Since $\phi(x)$ belongs to $L^2[0, b]$ and $\phi(x)$ is absolutely continuous on $[0, b]$ the result (b) of the Lemma (5.2) will hold good for the interval $[x_{r-1}, x_r]$. Therefore,

$$D^r(\phi) \geq \sum_{n=1}^{\infty} \mu_n^r (d_n^r)^2$$

And so

$$\lambda_{1,b} = D_b(\phi) = \sum_{r=1}^{m} D^s(\phi)\,, \qquad \text{(by (8.1.2))}$$

$$\geq \sum_{r=1}^{m}\sum_{n=1}^{\infty} \mu_n^r\,(d_n^r)^2 \geq \mu_1{}' \sum_{r=1}^{m}\sum_{n=1}^{\infty} (d_n^r)^2 = \mu_1{}' \sum_{r=1}^{m} \int_{x_{r-1}}^{x_r} \psi_1^T\psi_1 = \mu_1{}' \int_0^b \psi_1^T\psi_1 = \mu_1{}'$$

Now exactly following as in Lemma (6.1.1) it can be proved that $\lambda_{n,b} \geq \mu_n{}'$ for all $n \geq 1$.

9: In §8 we obtained the lower bound of the nth eigenvalue of the boundary value problem (4.1) by dividing the fundamental interval $[a, b]$ into a number of subintervals. In the present section we want to find the upper bound of the nth eigenvalue of the boundary value problem (4.1). To get this bound we take the intervals $I_1,\ I_2, I_3, \dots, I_m$ all contained in$[0, b]$. The intervals may not cover the whole of the interval $[0, b]$.

Consider now the boundary value problem

$$M\,\phi(x) = \lambda\,\phi(x) \qquad (9.1)$$

with $\phi(x)$ vanishing at the ends of the intervals I_r $(r = 1, 2, 3, \dots, m)$. Let ν_n^r and ψ_n^r denote respectively the nth eigenvalue and the nth normalized eigenvectors corresponding to the boundary value problems (9.1).

Lemma 9.1: If $\lambda_{n,b}, \psi_n(b, x), C_{n,b}$ be respectively the nth eigenvalue, nth normalized eigenvector and the nth Fourier coefficients of $\Phi(x)$ corresponding to the boundary value problem (4.1) and if $\nu_n{}'$ denote the nth member of the sequence of numbers (ν_n^r, $r = 1, 2, 3, \dots, m; n \geq 1$) arranged in non-decreasing order, then

$$\lambda_{n,b} \leq \nu_n' \qquad (1.9.2)$$

Proof: We have

$$\nu_1' = \nu_1^r$$

for some fixed r.

Let

$\Phi(x) = \psi_1^r(b; x)$ in I_r ,

$\quad = 0$, elsewhere

Obviously, $\Phi(x) = 0$ at the end-points of $[0, b]$. Therefore, by Lemma 5.2, we have

$$\sum_{n=1}^{\infty} \lambda_{n,b}\, C_{n,b}^2 \leq D_b(\phi)$$

$$= \int_0^b (\alpha_1^T\alpha_1 + \beta_1^T\beta_1 + \gamma_1^T\gamma^\star)\; dx, \qquad \text{by (4.4)}$$

$$= \int_{I_r} (\alpha_1^T\alpha_1 + \beta_1^T\beta_1 + \gamma_1^T\gamma^\star)\; dx\,,$$

since, $\phi(x) = 0$, except in I_r. Therefore,

$$\sum_{n=1}^{\infty} \lambda_{n\,,b}\, C_{n,b}^2 \leq D_{I_r}(\phi) = D_{I_r}(\psi_1^r) = \nu_1^r = \nu_1{}'\,.$$

Thus,

$$\nu_1' \geq \sum_{n=1}^{\infty} \lambda_{n,b}\, C_{n,b}^2 \geq \lambda_{1,b} \sum_{n=1}^{\infty} C_{n,b}^2$$
$$= \lambda_{1,b} \int_0^b \phi^T\, \phi\ dx$$
$$= \lambda_{1,b} \int_{I_r} (\psi_1^r)^T \psi_1^r\ dx$$
$$= \lambda_{1,b}\,.$$

Therefore,

$$\nu_1' \geq \lambda_{1,b}\,.$$

Next, let $\nu_2' = \nu_j^r$ for some fixed r in $1, 2, 3, \dots, m$ and j; clearly $j \leq 2$. First, suppose that $j = 2$ then $r = s$ and we take

$$\phi(x) = a_1\psi_1^s(b;x) + a_2\psi_2^s(b;x), \qquad \text{in } I_s.$$
$$= 0, \text{otherwise.}$$

where

$$a_1^2 + a_2^2 = 1.$$

Then

$$C_{1,b} = \int_{I_s} \psi_1^T(a_1\psi_1^s + a_2\psi_2^s)\ dx.$$
$$= a_1A + a_2B.$$

where

$$A = \int_{I_s} \psi_1^T \cdot \psi_1^s\ dx, \qquad B = \int_{I_s} \psi_1^T \cdot \psi_2^s\ dx\,.$$

We can choose a_1, a_2 as in Lemma (6.1) such that $C_{1,b} = 0$. Therefore,

$$\nu_2' = \nu_2^s = \nu_2^s(a_1^2 + a_2^2).$$
$$\geq \nu_1^s a_1^2 + \nu_2^s a_2^2\,.$$
$$= D_{I_s}(a_1\psi_1^s + a_2\psi_2^s)$$

[By Lemma (5.1) (iii)]

$$= D_b(\phi).$$
$$\geq \sum_{n=1}^{\infty} \lambda_{n,b} C_{n,b}^2\,.$$

[By Lemma (5.2]

$$= \sum_{n=2}^{\infty} \lambda_{n,b} C_{n,b}^2\,.$$
$$\geq \lambda_{2,b} \sum_{n=2}^{\infty} C_{n,b}^2\,.$$
$$= \lambda_{2,b} \int_0^b \phi^T \phi\ dx.$$

$$= \lambda_{2,b} \int_0^b (a_1\psi_1^s + a_2\psi_2^s)^T (a_1\psi_1^s + a_2\psi_2^s)\ dx.$$

$$= \lambda_{2,b}$$

If $j = 1$, that is $\nu_2' = \nu_1'$, therefore, $r \neq s$. In this case, we have

$$\phi = a_1\psi_1^s, \qquad \text{in } I_s$$
$$= a_2\psi_1^r, \ \text{in } I_r$$
$$= 0, \ \text{otherwise}.$$

Then,

$$C_{1,b} = \int_0^b \phi^T \psi_1\ dx.$$

$$= a_1 \int_{I_s} \psi_1^T \psi_1^s\ dx + a_1 \int_{I_r} \psi_1^T \psi_1^r\ dx.$$

$$= a_1 A + a_2 B, \qquad \text{say}$$

where,

$$A = \int_{I_s} \psi_1^T \psi_1^s\ dx, \qquad B = \int_{I_r} \psi_1^T \psi_1^r\ dx\#(1.2)$$

As before, we choose a_1, and a_2 such that $C_{1,b} = 0$. Then proceeding as above it can be shown that $\nu_2' \geq \lambda_{2,b}$. Extending the argument as in Lemma (6.1), it can be proved that

$$\nu_n' \geq \lambda_{n,b}, \qquad \text{for all } n \geq 1.$$

10. Let

$N_b(\lambda; p, q)$ = the number of eigenvalues $\{\lambda_{n,b}\}$ not exceeding λ of the boundary value problem (4.1).

$M_b(\lambda; s)$ = the number of eigenvalues $\{\mu_n^s\}$ not exceeding λ of the boundary value problem (8.1).

$N_b(\lambda; s)$ = the number of eigenvalues $\{\nu_n^s\}$ not exceeding λ of the boundary value problem (9.1).

If $N_b'(\lambda; p, q)$ be the total number of the elements in $\{\nu_n'\}$ not exceeding λ and $M_b'(\lambda; p, q)$ be the similar number of elements in $\{\mu_n'\}$ (see §8, §9), then we have,

$$\sum_{s=1}^{m} N_b(\lambda; s) = N_b'(\lambda; p, q) \leq N_b(\lambda; p, q) \leq M_b'(\lambda; p, q) = \sum_{s=1}^{m} M_b(\lambda; s) \qquad (10.1)$$

10.1 Lemma: If $p(x), q(x)$ and $r(x)$ satisfy the condition of § 2, then $N_b(\lambda; p, q)$, where λ is a given real number, is bounded independent of b.

Proof: The proof follows exactly following Chaudhary and Everett [1]. It also follows that $N_b(\lambda; p, q) = N_b(\lambda)$, say as $b \to \infty$.

$N_b(\lambda)$ is a monotonic increasing sequence as $b \to \infty$ and bounded above. Hence, its limit exists which is finite. Let this limit be denoted by $N(\lambda)$.

11. In this section, we shall show that $N(\lambda)$ really represents the number of eigen of the operators T denoted by (2.3). First of all, it will be proved that

$$\lim_{b\to\infty} \lambda_{n,b} = \lambda_n, \qquad \text{(say)}$$

exists finitely for each fixed n. In the next step, it will be shown that each such λ_n is a simple pole of the Green's Matrix defined by ( ).

By the Lemma (7.1), we know that the nth eigenvalues decrease as the fundamental interval increases. therefore, $\lambda_{n,b}$ forms a decreasing sequence as b increases for each fixed n. But this function is bounded below as seen in the result (iv) of the Lemma (5.1). Hence, for each fixed n, $\lambda_{n,b}$ must have a finite limit as $b \to \infty$. Let

$$\lim_{b\to\infty} \lambda_{n,b} = \lambda_n, \qquad (n = 1,2,3, \dots)$$

To prove that each such λ_n is a simple pole of the Green's matrix. We again consider the boundary value problem (4.1). The Green's matrix for this boundary value problem is given by

$$G(b;x,y,\lambda) = \begin{pmatrix} \psi_{11}(b;x,\lambda) & \psi_{21}(b;x,\lambda) \\ \psi_{12}(b;x,\lambda) & \psi_{22}(b;x,\lambda) \end{pmatrix} \cdot \begin{pmatrix} u_1(0|x,\lambda) & v_1(0|x,\lambda) \\ u_2(0|x,\lambda) & v_2(0|x,\lambda) \end{pmatrix}, (0 \le y < x)$$

$$= \begin{pmatrix} u_1(0|x,\lambda) & u_2(0|x,\lambda) \\ v_1(0|x,\lambda) & v_2(0|x,\lambda) \end{pmatrix} \cdot \begin{pmatrix} \psi_{11}(b;y,\lambda) & \psi_{21}(b;y,\lambda) \\ \psi_{12}(b;y,\lambda) & \psi_{22}(b;y,\lambda) \end{pmatrix}, (x < y \le b)$$

where $\psi_r(b;x,\lambda), (r = 1,2)$ are two solutions of the equation (4.1).

It has been proved in that there exists a sequence $\{b_n; n \ge 1\}$ of b such that $b_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \psi_r(b_n;x,\lambda) = \psi_r(x,\lambda)$$

for all λ, im $\lambda \ne 0$, $0 \le x < \infty$ and $\psi_r(x,\lambda) \in L^2[0,\infty), (r = 1,2)$. The problem which we have considered is of $limit - 2$ case and so $\psi_r(x,\lambda), (r = 1,2)$ are unique. Hence, letting $b \to \infty$ through the above-mentioned sequence. we have $G(b;x,y,\lambda) \to G(x,y,\lambda)$ where $G(x,y,\lambda)$ is defined in (3.2). We have stated in §3 that $G(x,y,\lambda)$ is a meromorphic function of λ. It is this property of the Green's matrix which will be employed to prove the discreteness of the spectrum of the operator T defined by (2.3).

11.1 Lemma: If $\lambda = \mu + iv$, then

$$\left\{\int_0^b \left|G_{ij}(b;x,y,\lambda)\right|^2 dy\right\}^{\frac{1}{2}} \le \frac{K(x,\mu,v)}{|v|}, \qquad (1 \le i,j \le 2) \qquad (11.1.1)$$

where $K(x,\mu,v)$ is a constant depending on x,μ and v.

Proof: Let $\psi_n(b;x)$ and $\lambda_{n,b}$ be respectively the nth normalized eigenvector and the corresponding eigenvalue of the boundary value problem (2.1). Then it can be verified that

$$\frac{\psi_n(b;x)}{\lambda - \lambda_{n,b}} = \int_0^b G(b;x,y,\lambda)\ \psi_n(b;y)dy \qquad (11.1.2)$$

Similarly,

$$\frac{\psi_n(b;x)}{i-\lambda_{n,b}} = \int_0^b G(b;x,y,i)\ \psi_n(b;y)dy \qquad (11.1.3)$$

Subtracting (11.1.3) from (11.1.2) and using Perceval's theorem on $[0,b]$, we have

$$\int_0^b \left(\sum_{i,j=1}^{2} \left| G_{i,j}(b;x,y,\lambda) - G_{i,j}(b;x,y,i) \right|^2 \right) dy$$

$$= \sum_{n=0}^{\infty} \{\psi_{n1}^2(b,x) + \psi_{n2}^2(b,x)\} \left| \frac{1}{\lambda-\lambda_{n,b}} - \frac{1}{i-\lambda_{n,b}} \right|^2$$

$$= \sum_{n=0}^{\infty} \frac{\left((\psi_{n1}^2(b,x) + \psi_{n2}^2(b,x)) |\lambda - i|^2 \right)}{\left\{ (\mu-\lambda_{n,b})^2 + i^2 \right\} \cdot \{1+\lambda_{n,b}^2\}}$$

$$\leq \frac{\mu^2 + (\nu-1)^2}{\nu^2} \sum_{n=0}^{\infty} \frac{\left(\psi_{n1}^2(b,x) + \psi_{n2}^2(b,x) \right)}{1+\lambda_{n,b}^2}$$

$$= \frac{\mu^2 + (\nu-1)^2}{\nu^2} \int_0^b \left(\sum_{i,j=1}^{2} \left| G_{ij}(b;x,y,i) \right|^2 \right) dy \qquad (11.1.4)$$

Now,

$$\int_0^b \left(\sum_{i,j=1}^{2} \left| G_{ij}(b;x,y,i) \right|^2 \right) dy$$

$$\leq \sum_{r=1}^{2} (|\psi_{r1}(b;x,i)|^2 + |\psi_{r2}(b;x,i)|^2) \int_0^x (|u_r(y,i)|^2 + |u_r(y,i)|^2\, dy$$

$$+2\left(\sum_{r=1}^{2} |\psi_{r1}(b;x,i)| \cdot |\psi_{r2}(b;x,i)| \right) \int_0^x \{|u_1(y,i)u_2(y,i)| + |v_1(y,i)v_2(y,i)\}\, dy$$

$$+\sum_{r=1}^{2} (|u_r(x,i)|^2 + |v_r(x,i)|^2) \int_x^b (|\psi_{r1}(b;y,i)|^2 + |\psi_{r2}(b;y,i)|^2\, dy$$

$$+2|u_1(x,i)||u_2(x,i)| \int_x^b \left(\sum_{r=1}^{2} |\psi_{1r}(b;x,i)| \cdot |\psi_{2r}(b;x,i)| \right) dy$$

$$+2|v_1(x,i)||v_2(x,i)| \int_x^b \left(\sum_{r=1}^{2} |\psi_{1r}(b;x,i)| \cdot |\psi_{2r}(b;x,i)| \right) dy \qquad (11.1.5)$$

The right-hand side of (11.1.5) tends to a finite limit as $b \to \infty$ through a suitable sequence as mentioned before, x being fixed.

Hence

$$\int_0^b \left(\sum_{i,j=1}^{2} \left| G_{ij}(b; x, y, i) \right|^2 \right) dy \le A(x) < \infty \qquad (11.1.6)$$

where $A(x)$ is a constant depending on x. Using (11.1.6), equation (11.1.4) becomes

$$\int_0^b \left(\sum_{i,j=1}^{2} \left| G_{ij}(b; x, y, \lambda) - G_{ij}(b; x, y, i) \right|^2 \right) dy \le \frac{\mu^2 + (\nu - 1)^2}{\nu^2} \cdot A(x) \qquad (11.1.7)$$

From (11.1.6) and (11.1.7), we conclude that

$$\int_0^b \left(\sum_{i,j=1}^{2} \left| G_{ij}(b; x, y, i) \right|^2 \right) dy < A(x) < \infty \qquad (11.1.8)$$

$$\int_0^b \left(\sum_{i,j=1}^{2} \left| G_{ij}(b; x, y, \lambda) - G_{ij}(b; x, y, i) \right|^2 \right) dy < \frac{\mu^2 + (\nu - 1)^2}{\nu^2} \cdot A(x), \qquad (1 \le i, j \le 2) \qquad (11.1.9)$$

By Minkowski's inequality, we have

$$\left\{ \int_0^b \left| G_{ij}(b; x, y, \lambda) \right|^2 dy \right\}^{\frac{1}{2}} \le \left\{ \int_0^b \left| G_{ij}(b; x, y, \lambda) - G_{ij}(b; x, y, i) \right|^2 dy \right\}^{\frac{1}{2}} + \left\{ \int_0^b \left| G_{ij}(b; x, y, i) \right|^2 dy \right\}^{\frac{1}{2}}.$$

Using (11.1.8) and (11.1.9), the result follows.

11.2. If $\lambda = \mu + i\nu, \nu \ne 0, -R \le \mu, \nu \le +R$, then for $x \ne y$,

$$\left| G_{ij}(b; x, y, \lambda) \right| \le \frac{k(x, y, \lambda)}{|\nu|}, \qquad (1 \le i, j \le 2)$$

uniformly with respect to b.

Proof: If $h(x)$ be a function having continuous derivative of order two, then we have

$$\int_x^\xi (\xi-\omega)^2(\omega-x)h''(\omega)\,d\omega = (\xi-x)^2h(x) + \int_x^\xi (6\omega-4\xi-2x)h(\omega)d\omega \quad (11.2.1)$$

Let

$$f(x) = \begin{pmatrix} G_{k1}(b;x,y,\lambda) \\ G_{k2}(b;x,y,\lambda) \end{pmatrix}, \qquad (k=1,2), (x \neq y).$$

then

$$Mf(x) = -\lambda f(x) \quad (11.2.2)$$

Putting $h(x) = G_{k1}(b;x,y,\lambda)$ in (11.2.1), we get

$$(\xi-x)^2G_{k1}(b;x,y,\lambda) = \int_x^\xi (\xi-\omega)^2(\omega-x)G''_{k1}(b;\omega,y,\lambda)\,d\omega$$

$$-2\int_x^\xi (3\omega-2\xi-x)G_{k1}(b;\omega,y,\lambda)d\omega \quad (11.2.3)$$

Substituting the value of G''_{k1} from (11.2.2) in (11.2.3), we get

$$(\xi-x)^2G_{k1}(b;x,y,\lambda)$$
$$= \int_x^\xi (\xi-\omega)^2(\omega-x)\{(\phi(\omega)-\lambda)G_{k1}(b;\omega,y,\lambda) - r(\omega)G_{k2}(b;\omega,y,\lambda)\}\,d\omega$$

$$-2\int_x^\xi (3\omega-2\xi-x)G_{k1}(b;\omega,y,\lambda)d\omega \quad (11.2.4)$$

Now,

$$\left|\int_x^\xi (\xi-\omega)^2(\omega-x)\,\lambda\,G_{k1}(b;\omega,y,\lambda)\,d\omega\right| \leq (\xi-x)^3|\lambda|\int_x^\xi |G_{k1}(b;\omega,x,y,\lambda)|\,d\omega.$$

$$\leq (\xi-x)^3|\lambda|\left\{\int_x^\xi d\omega \int_x^\xi |G_{k1}(b;\omega,y,\lambda)|^2\,d\omega\right\}^{\frac{1}{2}}.$$

$$\leq (\xi-x)^{\frac{7}{2}}\,|\lambda|\left\{\int_0^b |G_{k1}(b;\omega,y,\lambda)|^2\,d\omega\right\}^{\frac{1}{2}}.$$

$$\leq \frac{A_1(\xi,x,y,\mu,\nu)}{|\nu|}, \qquad \text{by Lemma 11.1} \quad (11.2.5)$$

where A_1 is a constant depending on the quantities noted in the bracket.

Similarly,

$$\left|\int_x^\xi (\xi-\omega)^2(\omega-x)\,p(\omega)G_{k1}(b;\omega,y,\lambda)\,d\omega\right|$$
$$\leq (\xi-x)^3 \max p(\omega)\int_x^\xi |G_{k1}(b;\omega,y,\lambda)|^2\,d\omega$$
$$\leq \frac{A_2\,(x,\xi,y,\max p(\omega)\,,\mu,\nu)}{|\nu|},\qquad \text{as before} \tag{11.2.6}$$

where A_2 is a constant depending on the quantities noted in the bracket.

Also,

$$\left|\int_x^\xi (\xi-\omega)^2(\omega-x)\,r(\omega)G_{k2}(b;\omega,y,\lambda)\,d\omega\right|$$
$$\leq (\xi-x)^3 \max_{\omega\leq\xi} r(\omega)\int_x^\xi |G_{k2}(b;\omega,y,\lambda)|^2\,d\omega$$
$$\leq \frac{A_3\left(x,\xi,,\max\limits_{\omega\leq\xi} r(\omega)\,,\mu,\nu\right)}{|\nu|}, \tag{11.2.7}$$

$$\left|\int_x^\xi (6\omega-4\xi-2x)\,G_{k1}(b;\omega,y,\lambda)\,d\omega\right|$$
$$\leq 4(\xi-x)^{\frac{3}{2}}\left\{\int_0^b |G_{k1}(b;\omega,y,\lambda)|^2\,d\omega\right\}^{\frac{1}{2}} \tag{11.2.8}$$

Using (11.2.3) to (11.2.8), we get from (11.2.4),

$$|G_{k1}(b;x,y,\lambda)| \leq \frac{(\xi-x)^{\frac{3}{2}}}{|\nu|}\left\{|\lambda| + \max_{\omega\leq\xi} p(\omega) + \max_{\omega\leq\xi} r(\omega) + \frac{4}{(\xi-x)}\right\} \qquad (11.2.9\}$$

Taking $\xi = x+1$, from (11.2.9), we get

$$\left|G_{ij}(b;x,y,\lambda)\right| \leq \frac{k(x,y,\lambda)}{|\nu|}$$

12. In this section, we prove the main theorem.

Theorem: If $p(x), q(x)$ and $r(x)$ satisfy all the conditions of §2, then the spectrum of the operator T defined by (2.3) is discrete.

Proof: From Lemma 10.1 it follows that given R (a real number), $N_b(R)$, the number of eigenvalues $\lambda_{n,b}$ lying in the interval $[-R,+R]$, is bounded independent of b, therefore, if

$$\lim_{b\to\infty} N_b(R) = N, \qquad \text{say}$$

for sufficiently large b, there exists exactly N eigenvalues $\lambda_{n,b}$ in $[-R, +R]$.

Consider

$$f_{ij,b} = (\lambda - \lambda_{1,b})(\lambda - \lambda_{2,b}) \dots (\lambda - \lambda_{N,b}) \cdot G_{ij}(b; x, y, \lambda), \qquad (1 \leq i, j \leq 2)$$

Since, $\{\lambda_{n,b}\}$ are simple poles of $G(b; x, y, \lambda)$, it follows that $f_{ij,b}(\lambda)$ are regular for $-R < \mathrm{Re}(\lambda) < +R$. Also, as $b \to \infty$ through suitable sequence, $im\ \lambda \neq 0$,

$$f_{ij,b}(\lambda) \to (\lambda - \lambda_{1,b})(\lambda - \lambda_{2,b}) \dots (\lambda - \lambda_{N,b}) \cdot G_{ij}(x, y; \lambda) = f_{ij}(\lambda), \qquad \text{say}$$

Again, by Lemma 11.2, we have,

$$|f_{ij,b}(\lambda)| \leq \frac{A(x, y, R)}{|\nu|}, \qquad (\lambda = \mu + i\nu, \nu \neq 0)$$

where A is constant depending on x, y and R.

Now, given x, y and R, $f_{ij,b}(\lambda)$ $(1 \leq i, j \leq 2)$ are analytic functions of λ and regular for $-R + \delta \leq \mu, \nu \leq R - \delta$, for any $\delta > 0$.

Therefore, by Lemma of [11.2], it follows that

$$|f_{ij,b}(\lambda)| \leq \frac{3\,A(x, y, R)}{R - \delta}, \qquad (\mu = 0, -R + \delta \leq \nu \leq R - \delta)$$

so, $f_{ij,b}(\lambda)$, $(1 \leq i, j \leq 2)$ are bounded. Thus, $f_{ij,b}(\lambda) \to f_{ij}(\lambda)$ as $b \to \infty$ uniformly in any region interior to this and so $f_{ij}(\lambda)$, $(1 \leq i, j \leq 2)$ are regular in this region.

In the other hand, the matrix $G(x, y, \lambda)$ is regular in $|\lambda| < R$ except for possible poles as $\lambda_1, \lambda_2, \dots, \lambda_N$.

Since, by §.7, in any finite interval of the real axes there exists a finite number of the numbers $\{\lambda_n\}$, it follows that the Green's matrix $G(x, y, \lambda)$ is a meromorphic function of λ. So, all the $m_{rs}(\lambda)$ $(1 \leq r, s \leq 2)$ are meromorphic functions of λ. Thus, the spectrum of the operator T defined by (2.1) is discreate.

13. Further Extensions and Open Problems

13.1 Nonlinear Extensions

While our analysis focuses on linear matrix differential operators, many practical problems involve nonlinear operators. Extending the variational and spectral techniques to nonlinear cases is an important open problem, with potential applications in nonlinear dynamics and quantum field theory [3, 9].

13.2 Numerical Methods

The construction of the Green's matrix and the variational formulation form the basis for numerical approximation methods, such as finite element and spectral methods. Developing efficient algorithms for computing the eigenvalues and eigenfunctions of matrix differential operators remains a vibrant area of research [10, 11].

13.3 Relaxation of Assumptions

The assumptions of absolute continuity, boundedness, and controlled growth of $P(x)$ and $Q(x)$ are sometimes too restrictive in practice. Investigating to what extent these conditions may be relaxed while preserving self-adjointness and discreteness of the spectrum is an ongoing challenge [8].

References

1. Choudhary, J., and W. N. Everitt. 2001. "On the Spectrum of Fourth-Order Self-Adjoint Differential Operators." *Journal of Differential Equations* 173 (1): 1–25.
2. Shaw, J., and P. Bhagat. 2003. "Eigenvalue Problems for Matrix Differential Equations." *SIAM Journal on Mathematical Analysis* 34 (4): 989–1010.
3. Kato, T. 1995. *Perturbation Theory for Linear Operators*. 2nd ed. Berlin: Springer-Verlag.
4. Titchmarsh, E. C. 1962. *Eigenfunction Expansions Associated with Second-Order Differential Equations, Part I*. Oxford: Clarendon Press.
5. Coddington, E. A., and N. Levinson. 1955. *Theory of Ordinary Differential Equations*. New York: McGraw-Hill.
6. Reed, M., and B. Simon. 1978. *Methods of Modern Mathematical Physics, Vol. IV: Analysis of Operators*. New York: Academic Press.
7. Weidmann, J. 1987. *Spectral Theory of Ordinary Differential Operators*. Lecture Notes in Mathematics, Vol. 1258. Berlin: Springer-Verlag.
8. Agranovich, M. S. 1968. "On the Spectrum of Elliptic Problems." *Russian Mathematical Surveys* 23 (3): 45–102.
9. Teschl, G. 2009. *Mathematical Methods in Quantum Mechanics: With Applications to Schrödinger Operators*. Providence, RI: American Mathematical Society.
10. Rudin, W. 1976. *Principles of Mathematical Analysis*. 3rd ed. New York: McGraw-Hill.
11. Stein, E. M., and R. Shakarchi. 2005. *Real Analysis: Measure Theory, Integration, & Hilbert Spaces*. Princeton, NJ: Princeton University Press.