VIRUS SPREAD IN AN INTUITIONISTIC FUZZY NETWORK BASED ON LAPLACIAN ENERGY

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ABSTRACT:
In this paper, we are discussed the virus spread in an intuitionistic fuzzy network and we defined the infection rate, curing rate and the sharp epidemic threshold of the virus spread in an intuitionistic fuzzy network and we illustrate these concepts with example.

AMS subject classification:
KEYWORDS: Intuitionistic fuzzy set, Virus spread, Epidemic threshold, Effective spreading rate.

1. INTRODUCTION

In this paper, we are discussed the virus spread in an intuitionistic fuzzy network. Each node(vertex) in this network is either infected or healthy. An infected node can infect its neighbors with an infection rate \(\beta\), and it is cured with curing rate \(\delta\). The ratio Fuzzy set was proposed by Lofti A. Zadeh in 1965 [17]. The generalization of fuzzy set, called intuitionistic fuzzy set was proposed by Atanassov in 1986 [2]. Graph theory was found by Leonhard Euler in 1735 when he considered the "Konigsberg bridges" problem. Graph theory having various applications like data mining, image segmentation, clustering, image capturing, communication networks, coding theory, group theory etc. The link structure of a webpage available at the website can be represented by an intuitionistic fuzzy directed graph in which the vertices are the links of the webpage available at the website and the directed edge from vertex A to vertex B exists if and only if A contains a link to B. In 1973, the definition of fuzzy graph was introduced by Kafmann [9] from the Zadeh’s fuzzy relations. In 1975, Rosenfeld[14] introduced another detailed definition including fuzzy vertex and fuzzy edges and several fuzzy analogs of graph theoretic concepts such as paths, cycles, connectedness and etc. In 1978, Gutman [7] introduced the energy of a graph as the sum of the absolute values of the eigen values of the adjacency matrix of a graph. The lower and upper bound for the energy of a graph are discussed in [5], [11], [8]. In 1999, the first definition of intuitionistic fuzzy graph was proposed by Atanassov [3]. The energy of fuzzy graph and its bounds are discussed in [1]. The energy of an intuitionistic fuzzy graph and its bounds are discussed in [13]. In [13], we defined the adjacency matrix of an intuitionistic fuzzy graph and we obtained the energy of an intuitionistic fuzzy graph. The lower and upper bound for the energy of an intuitionistic fuzzy graph are also obtained.

\[ \tau = \left( \frac{\beta}{\delta} \right) \]
then virus continue and a nonzero fraction of the nodes are infected, whereas $\tau \leq \tau_c$, the epidemic dies out [15], [16].

This paper is organized as follows. In section 2, we give all the necessary definitions and theorems related to the energy of an intuitionistic fuzzy graph. In section 3, we define the definition of infection rate and curing rate of the epidemic threshold of the virus spread in an intuitionistic fuzzy network. In section 4, we illustrate these concepts with example. In Section 5, we give the conclusion.

2. PRELIMINARIES


**Definition 2.1.** Let $G = (V,E)$ be a simple graph with $n$ vertices and $m$ edges. Let $A = [a_{ij}]$ be the adjacency matrix of $G$. Energy of a simple graph $G = (V,E)$ with adjacency matrix $A$ is defined as the sum of absolute values of eigenvalues of $A$. It is denoted by $E(G)$. That is,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_i$ is an eigenvalues of $A$, $i = 1, 2, \ldots, n$.

2.2. Energy of Fuzzy Graph [1]

**Definition 2.2.** Let $V$ be a nonempty set. A fuzzy subset of $V$ is a function $\sigma : V \rightarrow [0, 1]$. $\sigma$ is called the membership function and $\sigma (v)$ is called the membership of $v$ where $v \in V$. $V_1$ and $V_2$ be two nonempty sets and $\sigma_1$ and $\sigma_2$ be two fuzzy subsets of $V_1$ and $V_2$ respectively. Define a fuzzy subset $\mu$ of $V_1 \times V_2$ as $\mu(v_i,v_j) = \min(\sigma_1(v_i), \sigma_2(v_j))$. Then, $\mu$ is called a fuzzy relation, from $\sigma_1$ to $\sigma_2$. Suppose $\sigma_1(x) = 1, \forall x \in V_1$ and $\sigma_2(y) = 1, \forall y \in V_2$. Then $\mu$ is called a fuzzy relation from $V_1$ into $V_2$. $\mu(v_i,v_j)$ is interpreted as the strength of relation between $v_i$ and $v_j$. Suppose $V_1 \times V_2 = V$ and $\sigma_1 = \sigma_2 = \sigma$. Then $\mu$ is called a fuzzy relation on $V$. From the above definitions, it follows that binary relations on crisp sets are particular cases of fuzzy relations. Let $V$ be a nonempty set and $\sigma$ be a fuzzy subset of $V$. Let $\mu$ be a fuzzy relation on $\sigma$. $\mu$ is said to be symmetric, if $\mu(v_i,v_j) = \mu(v_j,v_i)$ for $v_i, v_j \in V$. A fuzzy relation can also be expressed by a matrix called fuzzy relation matrix $M = [m_{ij}]$, where $m_{ij} = \mu(v_i,v_j)$.

A fuzzy graph $G = (V, \sigma, \mu)$ is a nonempty set $V$ together with a pair of function ($\sigma$, $\mu$) where $\sigma$ is a fuzzy subset of $V$ and $\mu$ is a fuzzy relation on $\sigma$. Let $M = [m_{ij}]$ be a fuzzy relation matrix defined on $\mu$. This represents the strength of the relation between the vertices. Hence we have the following definitions.

The adjacency matrix $A$ of a fuzzy graph $G = (V, \sigma, \mu)$ is an $n \times n$ matrix where $n = |V|$ defined as $A = [a_{ij}]$, where $a_{ij} = \mu(v_i,v_j)$. Note that $A$ becomes the usual adjacency matrix when all the nonzero membership values are 1. i.e. when the fuzzy graph becomes a crisp graph. Let $G = (V, \sigma, \mu)$ be a fuzzy graph and $A$ be its adjacency matrix. Energy of $G$ is defined as the sum of the absolute values of the eigenvalues of $A$. 
2.3. Energy of an intuitionistic Fuzzy Graph [13]
In [13], we defined the adjacency matrix of an intuitionistic fuzzy graph and we obtained
the energy of an intuitionistic fuzzy graph. The link structure of a website could be represented
by
an intuitionistic fuzzy directed graph. The links are considered as vertices and the path
between
the links are considered as edges. The weightage of the each edge are considered as the number
of visitors (membership value), the number of non-visitors (non-membership value) and drop
off (intuitionistic fuzzy index) among that link structure.

**Definition 2.3.** An intuitionistic fuzzy graph is defined as $G = (V, E, \mu, \gamma)$ where $V$ is the set
of vertices and $E$ is the set of edges. $\mu$ is a fuzzy membership function defined on $V \times V$ and $\gamma$
is
a fuzzy non-membership function defined on $V \times V$. We denote $\mu(v_i, v_j)$ by $\mu_{ij}$ and $\gamma(v_i, v_j)$
by $\gamma_{ij}$ such that
(i) $0 \leq \mu_{ij} + \gamma_{ij} \leq 1$
(ii) $0 \leq \mu_{ij}, \gamma_{ij}, \pi_{ij} \leq 1$, where $\pi_{ij} = 1 - \mu_{ij} - \gamma_{ij}$. Hence $(V \times V, \mu, \gamma)$ is an intuitionistic fuzzy set.

![Figure 1: $G_1$, An intuitionistic fuzzy graph.](image-url)
Definition 2.5. An intuitionistic fuzzy adjacency matrix of an intuitionistic fuzzy graph is defined as the adjacency matrix of the corresponding intuitionistic fuzzy graph. That is for an intuitionistic fuzzy graph \( G = (V,E, \mu, \gamma) \), an intuitionistic fuzzy adjacency matrix is defined by \( A(IG) = [a_{ij}] \), where \( a_{ij} = (\mu_{ij}, \gamma_{ij}) \). Note that \( \mu_{ij} \) represents the strength of the relationship between \( v_i \) and \( v_j \) and \( \gamma_{ij} \) represents the strength of the non-relationship between \( v_i \) and \( v_j \).

Example 2.6. Laplacian Energy of an intuitionistic fuzzy graph in Fig. 1,

The adjacency matrix is
\[
A(IG) = \begin{bmatrix}
0 & (0.6,0.2) & 0 & (0.3,0.6) \\
(0.4,0.3) & 0 & (0.5,0.1) & 0 \\
(0.5,0.2) & (0.7,0.1) & 0 & (0.6,0.3) \\
(0.9,0.1) & (0.8,0.1) & (0.3,0.5) & 0
\end{bmatrix}
\]

Definition 2.7. The adjacency matrix of an intuitionistic fuzzy graph can be written as two matrices one containing the entries as membership values and the other containing the entries as non-membership values, i.e. \( A(IG) = (A(\mu),A(\gamma)) \), where \( A(\mu) = (\mu_{ij}) \) is the matrix representing the membership values and \( A(\gamma) = (\gamma_{ij}) \) is the matrix representing the non-membership values.

\[
A(\mu) = \begin{bmatrix}
0 & 0.6 & 0 & 0.3 \\
0.4 & 0 & 0.5 & 0 \\
0.5 & 0.7 & 0 & 0.6 \\
0.9 & 0.8 & 0.3 & 0
\end{bmatrix}
\]

\[
D(\mu) = \begin{bmatrix}
1.8 & 0 & 0 & 0 \\
0 & 2.1 & 0 & 0 \\
0 & 0 & 0.8 & 0 \\
0 & 0 & 0 & 0.9
\end{bmatrix}
\]

\[
L(\mu) = D(G) - A(G) = \begin{bmatrix}
1.8 & -0.6 & 0 & -0.3 \\
-0.4 & 2.1 & -0.5 & 0 \\
-0.5 & -0.7 & 0.8 & -0.6 \\
-0.9 & -0.8 & -0.3 & 0.9
\end{bmatrix}
\]

and
\[ A(\gamma) = \begin{bmatrix} 0 & 0.2 & 0 & 0.6 \\ 0.3 & 0.1 & 0 & 0 \\ 0.2 & 0.1 & 0 & 0.3 \\ 0.1 & 0.1 & 0.5 & 0 \end{bmatrix}, \quad D(\gamma) = \begin{bmatrix} 0.6 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \\ 0 & 0 & 0 & 0.9 \end{bmatrix} \]

\[ L(\gamma) = \begin{bmatrix} 0.6 & -0.2 & 0 & -0.6 \\ -0.3 & 0.4 & -0.1 & 0 \\ -0.2 & -0.1 & 0.6 & -0.3 \\ -0.1 & -0.1 & -0.5 & 0.9 \end{bmatrix} \]

**Definition 2.8.** The eigen values of an intuitionistic fuzzy adjacency matrix \( A(IG) \) is defined as \((X, Y)\) where \(X\) is the set eigen values of \(A(\mu)\) and \(Y\) is the set of eigen values of \(A(\gamma)\).

**Definition 2.9.** The energy of an intuitionistic fuzzy graph \( G = (V,E, \mu, \gamma) \) is defined as

\[
\left( \sum_{i \in x} |\lambda_i|, \sum_{i \in y} |\delta_i| \right)
\]

Where

\[
\sum_{i \in x} |\lambda_i|
\]

is defined as the energy of the membership matrix denoted by \(E(\mu(G))\) and

\[
\sum_{i \in y} |\delta_i|
\]

is defined as the energy of the non membership matrix denoted by \(E(\gamma(G))\).

**Example 2.10.** For an intuitionistic fuzzy graph in Fig. 1,

Eigen values =\{0.2, 5.172, 1.7558, 1.3270\}

\[
LE(\mu(G)) = \frac{\sum_{i=1}^{n} |\lambda_i - \frac{\sum(\mu)}{n}|}{n} = 2.8 + 0.2828 + 1.0442 + 1.473 = 5.6
\]

Eigen values = \{0, 1, 1, 0.5\}

\[
LE(\gamma(G)) = \left\{ \left| 0 - \frac{2.5}{4} \right| + \left| - \frac{2.5}{4} \right| + \left| - \frac{2.5}{4} \right| + \left| 0.5 - \frac{2.5}{4} \right| \right\} = 0.625 + 0.375 + 0.375 + 0.125
\]

\(LE\ (G_1) = (5.6, 1.5)\)

\(LE\ (\mu(G_1)) > LE(\gamma(G_1))\)
2.4. Epidemic threshold [16]

**Theorem 2.11. (Epidemic threshold)** The epidemic threshold \( \tau_c \) is
\[
\tau_c = \frac{1}{\lambda_{\max}(L)}
\]
where \( \lambda_{\max}(L) \) is the largest eigenvalue of the adjacency matrix \( L \) of the network.

We will prove this in two parts: the necessity of this condition in eliminating an infection and the sufficiency of this condition for wiping out any initial infection. The corresponding theorem statements are shown below. Following this, we will see how quickly an infection dies out if the epidemic threshold condition is satisfied.

**Theorem 2.12. (Necessity of epidemic threshold)** In order to ensure that over time, the infection probability of each node in the graph goes to zero (that is, the epidemic dies out), we must have
\[
\frac{\beta}{\delta} < \tau_c = \frac{1}{\lambda_{\max}(L)},
\]
where \( \beta \) is the infection rate, \( \delta \) is the curing rate and \( \lambda_{\max}(L) \) is the largest eigenvalue of the adjacency matrix \( L \).

**Theorem 2.13. (Sufficiency of epidemic threshold)**

If \( \frac{\beta}{\delta} < \tau_c = \frac{1}{\lambda_{\max}(L)} \), then the epidemic will die out over time (the infection probabilities will go to zero), irrespective of the size of the initial outbreak of infection.

3. VIRUS SPREAD IN AN INTUITIONISTIC FUZZY NETWORK

In this section, we consider an intuitionistic fuzzy network \( G = (V,E, \mu, \gamma) \). Let \( L(I G) = (L(\mu), L(\gamma)) \) be the Laplacian matrix of an intuitionistic fuzzy network. Let \( \beta \) be the infection rate and \( \delta \) be the curing rate of an intuitionistic fuzzy network. The ratio \( \delta = \frac{\beta}{\delta} \) is an intuitionistic fuzzy effective spreading rate and \( \tau_c \) is the epidemic threshold of an intuitionistic fuzzy network. If the effective spreading rate \( \tau = \left( \frac{\beta}{\delta} \right) > \tau_c \), then virus continue and a nonzero fraction of the nodes are infected, whereas \( \tau \leq \tau_c \), the epidemic dies out. We define the following definitions.

**Definition 3.1.**

If \( L(E(\mu(G))) > L(E(\gamma(G))) \)

i.e. If the number of visitors are maximum in an intuitionistic fuzzy network then the spreading rate of virus will be maximum. Otherwise we can say the system or network is unstable.
Definition 3.2.

If \( L(E(\mu(G))) \gg L(E(\gamma(G))) \)

then the infection rate of an intuitionistic fuzzy network is defined as \( \beta = \max_{i,j} \mu_{ij} \) and the curing rate of an intuitionistic fuzzy network is defined as \( \delta = \min_{i,j} \gamma_{ij} \). The ratio \( \tau = \frac{\beta}{\delta} \) (\( \delta \neq 0 \)) is the effective spreading rate and \( \tau_c = \frac{1}{\lambda_{\max} L(\mu)} \) where \( \lambda_{\max} L(\mu) \) is the largest eigen value of the Laplacian matrix \( L(\mu) \) of an intuitionistic fuzzy network.

Definition 3.3.

If \( L(E(\mu(G))) \ll L(E(\gamma(G))) \)

i.e. If the number of visitors are minimum in an intuitionistic fuzzy network then the spreading rate of virus will be minimum. Otherwise we can say the system or network is stable.

Definition 3.4.

If

\( L(E(\mu(G))) \ll L(E(\gamma(G))) \)

then the infection rate of an intuitionistic fuzzy network is defined as \( \beta = \min_{i,j} \mu_{ij} \) and the curing rate of an intuitionistic fuzzy network is defined as \( \delta = \max_{i,j} \gamma_{ij} \). The ratio \( \tau = \frac{\beta}{\delta} \) is the effective spreading rate and \( \tau_c = \frac{1}{\lambda_{\max} L(\gamma)} \) where \( \lambda_{\max} L(\gamma) \) is the largest eigen value of the Laplacian matrix \( L(\gamma) \) of an intuitionistic fuzzy network.

4. NUMERICAL EXAMPLES

Example 4.1. In example 2.4,

\( L(E(\mu(G))) \gg L(E(\gamma(G))) \)
\[ \beta = \max_{i,j} |\mu_{ij}| = 2.1 \]

\[ \delta = \min_{i,j} |\nu_{ij}| = 0.3 \]

\[ \tau = \frac{\beta}{\delta} = \frac{2.1}{0.3} = 7 \]

\[ \tau_c = \frac{1}{\lambda_{\max} L(\mu)} = \frac{1}{2.5172} = 0.3972 \]

Here \( \tau > \tau_c \), so the virus continue and a nonzero fraction of the nodes are infected. Hence the spreading rate of virus will be maximum.

**Example 4.2.** For an intuitionistic fuzzy graph in Fig. 2,

![Intuitionistic fuzzy graph](image)

**Figure 2:** \( G_2 \), An intuitionistic fuzzy graph.

**Example 4.3.** In example 4.2,
The adjacency matrix is

$$A (IG) = \begin{bmatrix}
0 & (0.1, 0.1) & 0 & (0.0, 0.2) \\
(0.2, 0.5) & 0 & (0.1, 0.7) & 0 \\
(0.0, 0.5) & (0.2, 0.3) & 0 & (0.1, 0.4) \\
(0.0, 0.6) & (0.0, 0.6) & (0.0, 0.6) & 0 \\
\end{bmatrix}$$

$$A (\mu) = \begin{bmatrix}
0 & 0.1 & 0 & 0 \\
0.2 & 0 & 0.1 & 0 \\
0 & 0.2 & 0 & 0.1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}$$

$$D (\mu) = \begin{bmatrix}
0.2 & 0 & 0 & 0 \\
0 & 0.3 & 0 & 0 \\
0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0.1 \\
\end{bmatrix}$$

$$L (\mu) = D(G) - A(G) = \begin{bmatrix}
0.2 & -0.1 & 0 & 0 \\
-0.2 & 0.3 & -0.1 & 0 \\
0 & -0.2 & 0.1 & -0.1 \\
0 & 0 & 0 & 0.1 \\
\end{bmatrix}$$

Eigen values =\{0.4414, 0.1586, -0.0000, 0.1000\}

$$LE[\mu(G)] = \sum_{i=1}^{n} \lambda_i - \frac{\sum(\mu_i)}{n} = \left| 0.4414 - \frac{0.7}{4} \right| + \left| 0.1586 - \frac{0.7}{4} \right| + \left| -0.0000 - \frac{0.7}{4} \right| + \left| 0.1000 - \frac{0.7}{4} \right|$$

$$= 0.2644 + 0.0164 + 0.175 + 0.075$$

$$= 0.5328$$

$$L(E(\mu(G_2))) \prec L(E(\gamma(G_2)))$$

$$\beta = \max_{i,j} |\mu_{i,j}| = 0.3$$

$$\delta = \min_{i,j} |\gamma_{i,j}| = 0.1$$

$$\tau = \frac{\beta}{\delta} = \frac{0.3}{0.1} = 3$$

$$\tau_c = \frac{1}{\lambda_{\max}(L(\mu))} = \frac{1}{0.4414} = 2.2655$$

$$\tau > \tau_c \text{ i.e. } 3 > 2.2655$$
\[ A(\gamma) = \begin{bmatrix} 0 & 0.1 & 0 & 0.2 \\ 0.5 & 0 & 0.7 & 0 \\ 0.5 & 0.3 & 0 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0 \end{bmatrix}, \quad D(\gamma) = \begin{bmatrix} 1.6 & 0 & 0 & 0 \\ 0 & 1.0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 \\ 0 & 0 & 0 & 0.6 \end{bmatrix} \]

\[ L(\gamma) = D(\gamma) - A(\gamma) = \begin{bmatrix} 1.6 & -0.1 & 0 & -0.2 \\ -0.5 & 1.0 & -0.7 & 0 \\ -0.5 & -0.3 & 1.3 & -0.4 \\ -0.6 & -0.6 & -0.6 & 0.6 \end{bmatrix} \]

Eigen values = \{0, 1, 2, 2\}

\[ LE(\gamma(G)) = \left\{ 0 - \frac{0.7}{4} + \left| -\frac{0.7}{4} \right| + 2 \left| -\frac{0.7}{4} \right| + 2 \left| -\frac{0.7}{4} \right| \right\} \]

= 0.175 + 0.825 + 1.825 + 1.825 = 4.65

LE (G_1) = (0.5328, 4.65)
LE (\mu(G_1)) > LE(\gamma (G_1))

Here \( \tau < \tau_c \), so the epidemic dies out. Hence the spreading rate of virus will be minimum.

5. CONCLUSION
In this paper, we are discussed the virus spread in an intuitionistic fuzzy network based on Laplacian energy and we defined the infection rate, curing rate and the sharp epidemic threshold of the virus spread in an intuitionistic fuzzy network and we illustrate these concepts with example.

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