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THE DIVISIBILITY OF DOUBLE MERSENNE JOIN MATRICES BY THE DOUBLE MERSENNE MEET MATRICES

[1]N.Elumalai &[2]R.Anuradha

[1]Associate Professor of Mathematics,A.V.C College(autonomous),

Mannampandal – 609305,Mayiladuthurai,Tamilnadu, India

[2]Assistant Professor of Mathematics,A.V.C College(autonomous),

Mannampandal – 609305,Mayiladuthurai,Tamilnadu, India

ABSTRACT

We define the double mersenne meet matrix and double mersenne join matrices separately.Also, we divide the double mersenne join matrices by the double mersenne meet matrices.We calculate the determinant,trace and inverse of double Mersenne Meet Matrices by using arithmetical functions.

KEYWORDS: Double Mersenne Meet,Double Mersenne Join,Divisibility

INTRODUCTION

Let S = $\{x_1, x_2, ..., x_n\}$ be a set of n positive integers with $x_1 < x_2 < ... < x_n$ and let $f : P \to \mathbb{C}$ be a complex valued function on Z_+ (i.e., arithmetic function).Let (x_i, x_j) denotes the greatest common divisor (gcd) of x_i and x_j and defines the $n \times n$ matrix $(S)_f$ by $((S)_f)_{ij}$ = $f(x_i, x_j)$.We refer to $(S)_f$ as the GCD Matrix on S with respect to f.The Set S is said to be gcd-closed if $(x_i, x_j) \in S$ whenever $x_i, x_j \in S$.The set S is said to be factor-closed if it contains every positive divisor of each $x_i \in S$.Clearly, a factor-closed set is always gcd– closed but the converse does not hold.

This paper develops the divisibility of Meet and Join Matrices on the Posets.We

present a characterization for the matrix divisibility of the join Matrix by the Meet Matrix in the ring Z^{nXn} in terms of the usual divisibility in Z,where S is a Meet Closed set and f is an integer-valued function on P.K.Bourque and S.Ligh [1,2] ,S.Hong [5,6,7,] studied this subject extensively.P.Haukkanen and I.Korkee [3] and C.He and I.Zhao [4] are also developed in this divisibility.

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2.STRUCTURE OF DOUBLE MERSENNE MEET AND DOUBLE MERSENNE JOIN MATRICES

2.1 Definition:

A number is said to be S-Prime if it can be written in the form 4n+1.

2.2 Definition:

Let S = $\{x_1, x_2, ..., x_n\}$ be a subset of P and the $n \times n$ matrix $(M)_P = (M_{ij})$ where

$M_{ij} = 2^{x_i \wedge x_j} - 1$, is called the Mersenne Meet Matrix on M.

2.3 Definition:

Let S = $\{x_1, x_2, ..., x_n\}$ be a subset of P and the $n \times n$ matrix $(MM)_P = (MM_{ij})$ where

$MM_{ij} = 2^{2^{x_i \wedge x_j} - 1} - 1$, is called the double Mersenne Meet Matrix on S.

2.4 Definition:

Let S = $\{x_1, x_2, ..., x_n\}$ be a set of distinct positive integers and $n \times n$ matrix $[M]_P = [M_{ij}] = \dfrac{(2^{x_i} - 1)(2^{x_j} - 1)}{(2^{x_i \wedge x_j} - 1)}$, is called the Mersenne Join Matrix On S.

2.5 Definition:

Let S = $\{x_1, x_2, ..., x_n\}$ be a set of distinct positive integers and $n \times n$ matrix $[MM]_P = [MM_{ij}] = \dfrac{\left(2^{2^{x_i - 1}} - 1\right)\left(2^{2^{x_j - 1}} - 1\right)}{\left(2^{2^{x_i \wedge x_j - 1}} - 1\right)}$, is called the Double Mersenne Join Matrix On S.

2.6 Definition:

Let S = $\{x_1, x_2, ..., x_n\}$ be a set of distinct positive integers and $n \times n$ matrix R = (r_{ij}) where $r_{ij} = \dfrac{[M]_P}{(M)_P} = \dfrac{(2^{x_i} - 1)(2^{x_j} - 1)}{(2^{x_i \wedge x_j} - 1)^2}$ call it to be the Mersenne Join Matrix – Reciprocal Mersenne Meet Matrix on S.

2.7 Definition:

Let S = $\{x_1, x_2, ..., x_n\}$ be a set of distinct positive integers and $n \times n$ matrix Q = (q_{ij}) where $q_{ij} = \dfrac{[MM]_P}{(MM)_P} = \dfrac{\left(2^{2^{x_i - 1}} - 1\right)\left(2^{2^{x_j - 1}} - 1\right)}{\left(2^{2^{x_i \wedge x_j - 1}} - 1\right)^2}$ call it to be the Double Mersenne Join Matrix – Reciprocal Double Mersenne Meet Matrix on S.

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3.MAIN RESULTS

3.1 Theorem

Define $n \times n$ matrix $\Lambda = diag(g(x_1), g(x_2), \ldots, g(x_n))$, where $g(x_i) = \frac{1}{\left(2^{2^{x_i-1}}-1\right)^2} \sum_{x_j \le x_i} \left(2^{2^{x_j-1}}-1\right)^2 \mu(x_i, x_j)$ and $n \times n$ matrix E = (e_{ij}) by

$$e_{ij} = \begin{cases} 2^{2^{x_i-1}} & if \;\; 2^{2^{x_i-1}}-1 \Big/ 2^{2^{x_j-1}}-1 \\ 0 & otherwise \end{cases}$$ then Q = E Λ E^T

Proof:

The ij-entry in E Λ E^T is $(E\Lambda E^T)_{ij} = \sum_{k=1}^{n} e_{ik} g(x_k) e_{jk}$

$$= \sum_{\substack{x_k \le x_i \\ x_k \le x_j}} \left(2^{2^{x_i-1}}-1\right) g(x_k) \left(2^{2^{x_j-1}}-1\right)$$

$$= \left(2^{2^{x_i-1}}-1\right)\left(2^{2^{x_j-1}}-1\right) \sum_{x_k \le [x_i \vee x_j]} g(x_k)$$

Where g is an arithmetical functions.
By the Mobius Inversion formula, we have

$$\sum_{d/n} g(d) = \frac{1}{n^2}$$

$$\therefore \; (E\Lambda E^T)_{ij} = \frac{\left(2^{2^{x_i-1}}-1\right)\left(2^{2^{x_j-1}}-1\right)}{\left(2^{2^{x_i \wedge x_j}-1}-1\right)^2} = q_{ij}$$

3.2 Theorem

If Q is an $n \times n$ double Mersenne Join Matrix – Reciprocal Double Mersenne Meet Matrix on S then $\det(Q) = \prod_{i=1}^{n} \left(2^{2^{x_i-1}}-1\right)^2 g(x_i)$ where

$$g(x_i) = \frac{1}{\left(2^{2^{x_i-1}}-1\right)^2} \sum_{x_j \le x_i} \left(2^{2^{x_j-1}}-1\right) \mu(x_i, x_j)$$

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Proof:

By using the theorem (3.1),Q = E Λ E^T where $\Lambda = diag(g(x_1), g(x_2),..., g(x_n))$ and E = (e_{ij}) lower-triangular matrix with diagonal $\left(2^{2^{x_1-1}}-1, 2^{2^{x_2-1}}-1, 2^{2^{x_3-1}}-1,\ldots\ldots, 2^{2^{x_n-1}}-1\right)$ and det Q = $\prod_{k=1}^{n}\left(2^{2^{x_k-1}}-1\right)$

From these it follows that

$$\begin{aligned}\det\ Q &= (\det E)(\det\Lambda)(\det E^T)\\ &= (\det E)^2(\det\Lambda)\\ &= \prod_{k=1}^{n}\left(2^{2^{x_i-1}}-1\right)^2 g(x_i)\end{aligned}$$

3.3 Theorem

If Q = (q_{ij}) is an $n\times n$ double Mersenne Join-Matrix – Double Reciprocal Mersenne Meet Matrix on S then trace(Q) = n.

Proof:

The ij-entry of Q is q_{ij} where $q_{ij} = \dfrac{\left(2^{2^{x_i-1}}-1\right)\left(2^{2^{x_j-1}}-1\right)}{\left(2^{2^{x_i\wedge x_j-1}}-1\right)^2}$

Trace(Q) = $\sum_{i=1}^{n} r_{ii} = \sum_{i=1}^{n}\dfrac{\left(2^{2^{x_i-1}}-1\right)\left(2^{2^{x_j-1}}-1\right)}{\left(2^{2^{x_i\wedge x_j-1}}-1\right)^2} = \sum_{i=1}^{n} 1 = n$

3.4Theorem:

Let E = (e_{ij}) where $e_{ij} = \begin{cases} 2^{2^{x_i-1}}-1 & if \quad \dfrac{2^{2^{x_i-1}}-1}{2^{2^{x_j-1}}-1} \\ 0 & otherwise\end{cases}$

Let U = (u_{ij}) be defined by

$$u_{ij} = \begin{cases}\dfrac{1}{2^{2^{x_j-1}}-1}\mu\left(\dfrac{2^{2^{x_i-1}}-1}{2^{2^{x_j-1}}-1}\right) & if \quad \dfrac{2^{2^{x_i-1}}-1}{2^{2^{x_j-1}}-1} \\ 0 & otherwise\end{cases} \quad is\ inverse\, of\ E.$$

Proof:

The ij-entry of EU is $(EU)_{ij}$

$$(EU)_{ij} = \sum_{k=1}^{n} e_{ik}u_{kj} = \sum_{\substack{x_k/x_i\\ x_j/x_k}} (2^{2^{x_i-1}}-1)\frac{1}{2^{2^{x_j-1}}-1}\mu\left(\frac{2^{2^{x_i-1}}-1}{2^{2^{x_i-1}}-1}\right)$$

$$= \frac{2^{2^{x_i-1}}-1}{2^{2^{x_j-1}}-1} \sum_{x_d / \frac{x_i}{x_j}} \mu(x_d) = \begin{cases} 1 & if \quad x_i = x_j \\ 0 & otherwise \end{cases}$$

Thus $E^{-1} = U$.

3.5 Theorem:

If Q is invertible then the inverse of Q is the $n \times n$ matrix B = (b_{ij}) where

$$b_{ij} = \frac{1}{q(x_i)q(x_j)} \sum_{[x_i \vee x_j] \leq x_k} \frac{\mu(x_i / x_k)\mu(x_j / x_k)}{g(x_k)}$$

Proof:

By using the theorem (3.1), Q = $E \Lambda E^T \Rightarrow Q^{-1} = (E^T)^{-1} \wedge^{-1} E^{-1}$

By using the theorem(3.1) and (3.4),

$\Lambda = diag(g(x_1), g(x_2), ..., g(x_n))$ and U = E^{-1}, we have $Q^{-1} = U^T \Lambda^{-1} U = (b_{ij}) = B.$

Thus the proof of the theorem.

3.6Example:

Construct the 2×2 Double Mersenne Join Matrix – Double reciprocal Mersenne Meet Matrix on the lower closed –upper closed set S = {1,2},then by using the definition(2.7),

$$q_{ij} = \frac{\left(2^{2^{x_i-1}}-1\right)\left(2^{2^{x_j-1}}-1\right)}{\left(2^{2^{x_i \wedge x_j}-1}-1\right)^2}.$$

$$q_{11} = \frac{\left(2^{2^{1-1}}-1\right)\left(2^{2^{1-1}}-1\right)}{\left(2^{2^{1\wedge 1}-1}-1\right)^2} = \frac{1}{1} = 1$$

$$q_{12} = q_{21} = \frac{\left(2^{2^{1-1}}-1\right)\left(2^{2^{2-1}}-1\right)}{\left(2^{2^{1\wedge 2}-1}-1\right)^2} = \frac{1.7}{1} = 7$$

$$q_{22} = \frac{\left(2^{2^{2-1}}-1\right)\left(2^{2^{2-1}}-1\right)}{\left(2^{2^{2\wedge 2}-1}-1\right)^2} = \frac{7.7}{49} = 1$$

$$\therefore Q = \begin{bmatrix} 1 & 7 \\ 7 & 1 \end{bmatrix}$$

By using the theorem(3.2),

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$$\det(Q) = \prod_{i=1}^{n}\left(2^{2^{x_i}-1}-1\right)^2 g(x_i) \quad \text{where} \quad g(x_i) = \frac{1}{\left(2^{2^{x_i}-1}-1\right)^2}\sum_{x_j \le x_i}\left(2^{2^{x_j}-1}-1\right)\mu(x_i, x_j)$$

$$g(x_1) = g(1) = \frac{1}{\left(2^{2^1-1}-1\right)^2}\sum_{x_j \le 1}\left(2^{2^{x_j}-1}-1\right)^2 \mu(x_i, x_j)$$

$$= \frac{1}{1}\left(2^{2^1-1}-1\right)\mu(1,1) = 1$$

$$g(x_2) = g(2) = \frac{1}{\left(2^{2^2-1}-1\right)^2}\sum_{x_j \le 2}\left(2^{2^{x_j}-1}-1\right)^2 \mu(x_i, x_j)$$

$$= \frac{1}{49}\left[\left(2^{2^1-1}-1\right)^2\mu(1,1) + \left(2^{2^2-1}-1\right)^2\mu(1,2)\right] = \frac{1}{49}[1+49(-1)]$$

$$= \frac{-48}{49}$$

$$\text{Det(Q)} = \left(2^{2^1-1}-1\right)^2 g(1)\left(2^{2^2-1}-1\right)^2 g(2) = 49.\frac{-48}{49} = -48$$

By using the theorem(3.3), trace(Q) = $q_{11}+q_{22}$ = 1+1 = 2
By using the theorem(3.5), Q^{-1} = (b_{ij}) where

$$b_{ij} = \frac{1}{q(x_i)q(x_j)}\sum_{[x_i \vee x_j] \le x_k}\frac{\mu(x_i/x_k)\mu(x_j/x_k)}{g(x_k)}$$

$$b_{11} = \frac{1}{q(x_1)^2}\sum_{x_1/x_k}\frac{\mu(x_1/x_k)^2}{g(x_k)} = \frac{1}{q(1)^2}\sum_{x_1/x_k}\frac{\mu(1/x_k)^2}{g(x_k)}$$

$$= \frac{1}{1}\left[\frac{\mu(1/1)^2}{g(1)} + \frac{\mu(1/2)^2}{g(2)}\right]$$

$$= \frac{1}{1} + \frac{(-1)^2}{-48/49} = 1 - \frac{49}{48} = \frac{-1}{48}$$

$$b_{12} = \frac{1}{q(x_1)q(x_2)} \sum_{x_1 \vee x_2 / x_k} \frac{\mu(x_1/x_k)\mu(x_2/x_k)}{g(x_k)} = \frac{1}{q(1)q(2)} \sum_{1 \vee 2/x_k} \frac{\mu(1/x_k)\mu(2/x_k)}{g(x_k)}$$

$$= \frac{1}{7} \sum_{2/x_k} \frac{\mu(1/x_k)\mu(2/x_k)}{g(2)}$$

$$= \frac{1}{7}\left[\frac{-1.1}{-48/49}\right]$$

$$= \frac{49}{336} = \frac{7}{48}$$

$$b_{12} = \frac{1}{q(x_1)q(x_2)} \sum_{x_1 \vee x_2 / x_k} \frac{\mu(x_1/x_k)\mu(x_2/x_k)}{g(x_k)} = \frac{1}{q(1)q(2)} \sum_{1 \vee 2/x_k} \frac{\mu(1/x_k)\mu(2/x_k)}{g(x_k)}$$

$$= \frac{1}{7} \sum_{2/x_k} \frac{\mu(1/x_k)\mu(2/x_k)}{g(2)}$$

$$= \frac{1}{7}\left[\frac{-1.1}{-48/49}\right]$$

$$= \frac{7}{48}$$

$$b_{22} = \frac{1}{q(x_2)^2} \sum_{x_2/x_k} \frac{\mu(x_2/x_k)^2}{g(x_k)} = \frac{1}{q(2)^2} \sum_{x_2/x_k} \frac{\mu(2/x_k)^2}{g(x_k)}$$

$$= \frac{1}{1}\left[\frac{\mu(2/1)^2}{g(2)} + \frac{\mu(2/2)^2}{g(1)}\right]$$

$$= \left[\frac{1}{-48/49} + \frac{1}{1}\right]$$

$$= \left[\frac{-49}{48} + 1\right] = \frac{-1}{48}$$

$$\therefore \quad Q^{-1} = \begin{bmatrix} -\frac{1}{48} & \frac{7}{48} \\ \frac{7}{48} & -\frac{1}{48} \end{bmatrix}$$

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