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Semi Parallel and Weyl-Semi Parallel Hypersurface of Tachibana Manifold

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Key words :

Semi Parallel Hypersurface, Weyl-Semi Parallel Hypersurface.

Abstract

In this paper, we have studied semi parallel and Weyl semi parallel para-Sasakian hypersurface of a Tachibana manifold. We have prove that para-Sasakian hypersurface of a Tachibana manifold is semi-parallel if and only if it is totally umbilical with negative unit mean curvature. Further we have prove that such a hypersurface is Weyl-semi-parallel if and only if it is either η−Einstein manifold or semi-parallel. Some more results has been studied in this paper.

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1. Introduction

We consider an isometric immersion $f: M \to \tilde{M}$ and let '*h*' is second fundamental form and ∇ is Vander-Waerden Bortolotti connection on *M*, then J. Deprez defined the immersion to be semi-parallel if

$$R(X,Y)h = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})h = 0, \qquad (1.1)$$

holds for all vector field tangent to *M*. Here '*R*' is curvature tensor of connection ∇. In [1] and [2] J. Deprez studied semi-parallel immersion in real space form. In [3], Ü Lumiste has proved that a semi-parallel submanifold is the second order envelope of the family of parallel submanifold. In [4] hypersurfaces of sphere and hyperbolic space has been studied by F. Dillen. He proved that semi-parallel hypersurfaces are flat surfaces with parallel Weingarten endomorphism or rotation hypersurface of certain helices. In [5] A. C. Asperti, C. A. Lobos and F. Mercuri defined that submanifolds satisfying

$$R.h = L_h Q(g, h), \quad (1.2)$$

are called pseudoparallel. Here L_h is some function on submanifold.

In [6] C. Ozgur defined the submanifold to Weyl-semi parallel if they satisfy

$$C.h = 0, \quad (1.3)$$

where '*C*' denotes the Weyl conformal curvature tensor. Generalization of Weyl semi-parallel condition has been studied in [6]. Here submanifolds satisfying

$$C.h = L_h.Q(g, h), \quad (1.4)$$

has been studied.

In this paper we have studied semi-parallel, pseudo-parallel and Weyl semi-parallel para-Sasakian hypersurface of a Tachibana manifold.

2. Para-Sasakian Manifold

An (2n + 1)–dimensional differentiable manifold *M* is called almost para contact manifold if it admits an almost para contact structure (ϕ, ξ, η) consisting of a (1, 1) tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = I - \eta \otimes \xi, \quad (2.1)$$

$$\eta(\xi) = 1, \quad (2.2)$$

$$\eta.\phi = 0, \quad (2.3)$$

$$\phi(\xi) = 0. \quad (2.4)$$

Let '*g*' be compatible-Riemannian metric with (ϕ, ξ, η), i.e.

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.5)$$

or equivalently

$$g(\phi X, Y) = g(X, \phi Y), \tag{2.6}$$

and

$$g(X, \xi) = \eta(X), \tag{2.7}$$

where *X*, *Y* are arbitrary vector fields on *M* then *M* is called an almost para contact Riemannian manifold with an almost para contact Riemannian structure (ϕ, ξ, η, g).

An almost para contact Riemannian manifold is called para-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.8}$$

where $X, Y \in T(M)$, ∇ is Levi-Civita connection of the Riemannian metric.

From above equations, we can get

$$\nabla_X \xi = \phi(X) \tag{2.9}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X. \tag{2.10}$$

In an para-Sasakian manifold *M*, the curvature tensor *R*, the Ricci tensor *S* and Ricci map $'S$ satisfy

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$

$$S(Y, \xi) = -2n\eta(Y), \tag{2.12}$$

$$'S(\xi) = -2n\xi, \tag{2.13}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.14}$$

$$\eta(R(X, Y), \xi) = 0, \tag{2.15}$$

$$\eta(R(\xi, X)Y) = \eta(X)\eta(Y) - g(X, Y). \tag{2.16}$$

We also know that differentiable manifold *M* is called η Einstein manifold if

$$S(X, Y) = a.g(X, Y) + b\eta(X).\eta(Y), \tag{2.17}$$

where *a*, *b* are some functions on *M*.

3. Hypersurface of Tachibana Manifold

An (2n+2)−dimensional differentiable manifold $\tilde{M}$ is called an almost Tachibana manifold if there exist a tensor field '*J*' of type (1, 1), Riemannian metric G satisfying

$$J^2 \tilde{X} = -\tilde{X}, \tag{3.1}$$

$$G(\surd\tilde{X}, \surd\tilde{Y}) = G(\tilde{X}, \tilde{Y}), \quad (3.2)$$

$$(\tilde{\nabla}_{\tilde{X}} J)\tilde{Y} + (\tilde{\nabla}_{\tilde{Y}} J)\tilde{X} = 0, \quad (3.3)$$

where $\tilde{\nabla}$ is Levi-Civita connection on $\tilde{M}$, and $\tilde{X}, \tilde{Y}$, are arbitrary vector fields on $\tilde{M}$.

In almost Tachibana manifold, we have

$$N(X, Y) = -4J((\tilde{\nabla}_{\tilde{X}} J)\, \tilde{Y}). \quad (3.4)$$

An almost Tachibana manifold on which Nijenhuis tensor '*N*' vanishes is called Tachibana manifold [7].

In a Tachibana manifold, we have

$$(\tilde{\nabla}_{\tilde{X}} J)\tilde{Y} = 0. \quad (3.5)$$

Now, let us suppose that para-Sasakian manifold '*M*' is isometrically embedded into a Tachibana manifold '$\tilde{M}$'. Then by Gauss and Weingarten equation, we get

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \nabla_X Y + H(X, Y)N, \quad (3.6)$$

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \nabla_X Y + H(X, Y).N, \quad (3.7)$$

where '*H*' denotes second fundamental form and '*A*' is shape operator. We have

$$G(\tilde{X}, \tilde{Y}) = g(X, Y), \quad (3.8)$$

$$G(\tilde{X}, N) = 0, \quad (3.9)$$

$$G(N, N) = 1, \quad (3.10)$$

$$H(X, Y) = g(AX, Y). \quad (3.11)$$

Here *X* is restriction of $\tilde{X}$ form $\tilde{M}$ to *M* and *N* is normal vector field to *M*.

We also know that a submanifold is totally geodesic iff second fundamental form vanishes.

From now we assume that *M* is hypersurface of Tachibana manifold $\tilde{M}$.

Let us put

$$\surd N = \tilde{\xi}, \quad (3.12)$$

$$\surd\tilde{X} = \phi X - \eta(X).N, \quad (3.13)$$

where $\tilde{\xi}$ is extension of ξ from *M* to $\tilde{M}$. Differentiating equation (3.12) and (3.13) with respect to connection $\tilde{\nabla}$ along the direction $\tilde{Y}$ and using equation (3.6) and (3.7) and comparing tangential part, we can easily get

$$-\phi(AX) = \nabla_X \xi, \tag{3.14}$$

$$H(X,Y)\xi = (\nabla_X \phi)Y + \eta(Y).A(X). \tag{3.15}$$

Comparing equations (3.14) and (2.9), we get

$$AX = -X + \eta(AX)\xi + \eta(X)\xi. \tag{3.16}$$

Contracting above equation

$$tr(A) = -2n + \eta(A(\xi)). \tag{3.17}$$

Also from (3.16), we have

$$A(\xi) = \eta(A\xi).\xi = (tr(A) + 2n)\xi, \tag{3.18}$$

from (3.16) and (3.18), we get

$$A(X) = -X + (tr\, A + zn + 1)\,\eta(X).\xi, \tag{3.19}$$

i.e.

$$A(X) = -X + \lambda.\eta(X).\xi, \tag{3.20}$$

where

$$\lambda = tr(A) + zn + 1. \tag{3.21}$$

We have some more common results. The Weyl conformal curvature tensor of a (2n + 1)–dimensional Riemannian manifold is given as

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\{Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)R(X)$$

$$-g(X,Z)RY\} + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.22}$$

$$(R(X,Y).H)(U,V) = -H(R(X,Y)U,V) - H(U,R(X,Y)V), \tag{3.23}$$

$$(C(X,Y).H)(U,V) = -H(C(X,Y)U,V) - H(U,C(X,Y)V), \tag{3.24}$$

$$Q(g,h)(U,V;X,Y) = -H((X \wedge Y)U,V) - H(U,(X \wedge Y)V), \tag{3.25}$$

where *X*, *Y*, *U*, *V* are tangent vector fields to *M* and $X \wedge Y$ is an endomorphism defined as

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y. \qquad (3.26)$$

4. Properties of Semi-parallel, Pseudo-parallel and Weyl Semi-parallel Hypersurfaces

Theorem 4.1. Let *M* be a para-Sasakian hypersurface of a Tachibana manifold $\tilde{M}$. Then *M* is semi-parallel if and only if it is totally umbilical with negative unit mean curvature.

Proof. Let *M* is semi-parallel, then we have

$$R(X, H).H = 0$$

i.e. $$(R(X, Y).H)(U, V) = 0.$$

i.e. $$-g(R(X, Y)U, AV) - g(AU, R(X, Y)V) = 0.$$

Using (3.20) in above equation, we get

$$\lambda.\eta(V).g(R(X, Y)U, \xi) + \lambda.\eta(U).g(R(X, Y)V, \xi) = 0.$$

Put $U = \xi$ in above equation

$$\lambda.g(R(X, Y)V, \xi) = 0 \quad \Rightarrow \quad \lambda R(X, Y)\xi = 0$$

$$\Rightarrow \quad \lambda.S(Y, \xi) = 0 \quad \Rightarrow \quad \lambda.2n.\xi = 0$$

$$\Rightarrow \quad \lambda = 0,$$

then from equation (3.20), we get

$$AX = -X \quad \text{or} \quad A = -I$$

from above equation, we get

$$g(AX, Y) = -g(X, Y), \qquad \text{i.e.} \qquad H(X, Y) = -g(X, Y).$$

Therefore *M* is totally umbilical with negative unit mean curvature.

By reversing the above steps, we can easily get the converse part of the theorem.

Theorem 4.2. Let *M* is a para-Sasakian hypersurface of a Tachibana manifold $\tilde{M}$. Then *M* is pseudo-parallel with $L_H \neq 1$ if and only if it is totally umbilical with negative unit mean curvature.

Proof. From equation (3.25) and (3.26), we have

$$Q(g, H)(U, V, X, Y) = -g(Y, U)H(X, V) + g(X, U)H(Y, V)$$
$$- g(Y, V).H(X, U) + g(X, V)H(Y, U).$$

Using (3.20) in above, we get

$$Q(g,H)(U,V;X,Y)=\lambda[-g(Y,U)\eta(X)\eta(V)+g(X,U)\eta(Y)\eta(V)$$
$$-g(Y,V)\eta(X)\eta(U)+g(X,V)\eta(Y)\eta(U)].$$

Let the hypersurface *M* is pseudo-parallel, then we have

$$R.H=L_H.Q(g,H).$$

Using (4.2) in above equation, we get

$$\lambda[\eta(V).g(R(X,Y)U,\xi)+\eta(U).g(R(X,Y)V,\xi)]=\lambda L_H[-g(Y,U)\eta(X)\eta(V)$$
$$+g(X,U)\eta(Y)\eta(V)-g(Y,V)\eta(X)\eta(U)+g(X,V)\eta(Y)\eta(U)].$$

Put $U=\xi$ in above equation, we get

$$\lambda[L_H-1][-g(Y,V)\eta(X)+g(X,V)\eta(Y)]=0.$$

Since $L_H\neq 1$, $\therefore$ $\lambda=0$, then from equation (3.20), we get

$$AX=-X \quad \text{or} \quad A=-I \quad \Rightarrow \quad H(X,Y)=-g(X,Y)$$

which shows that hypersurface *M* is totally umbilical with negative unit mean curvature.

Conversely, let the hypersurface '*M*' is totally umbilical then by previous theorem it is semi-parallel. We also know that semi-parallel hypersurfaces are pseudo-parallel. This proves the converse part.

Theorem 4.3. Let *M* be a para-Sasakian hypersurface of a Tachibana manifold $\tilde{M}$. Then *M* is Weyl semi-parallel if and only if it is an $\eta-$Einstein manifold or it is totally umbilical with negative unit mean curvature.

Proof. Let the hypersurface *M* is Weyl semi-parallel, i.e.

$$C.H=0$$

i.e. $(C(X,Y).H)(U,V)=0$

i.e. $-H(C(X,Y)U,V)-H(U,C(X,Y)V)=0$

i.e. $g(C(X,Y)U,AV)+g(AU,C(X,Y)V)=0.$

Using (3.20) in above equation, we get

$$-g(C(X,Y)U,V)+\lambda\eta(V)g(C(X,Y)U,\xi-g(C(X,Y)V,U)$$
$$+\lambda\eta(U)g(C(X,Y)V,\xi)=0$$

$$\Rightarrow \quad \lambda[\eta(V)g(C(X,Y)U,\xi)+\eta(U)g(C(X,Y)V,\xi)]=0.$$

Put $U=\xi$ in above equation, we get

$$\lambda g(C(X,Y)V,\xi)=0, \tag{4.3}$$

then we have either $\lambda=0$ or $g(C(X,Y)V,\xi)=0$. If $\lambda=0$, then hypersurface *M* is totally umbilical with unit negative mean curvature and if $g(C(X,Y)V,\xi)=0$, then by equation (2.14) and (3.22), we get

$$0=\Big(1+\frac{r}{2n(2n-1)}\Big)(g(Y,V)\eta(X)-g(X,V)\eta(Y))$$
$$-\frac{1}{2n-1}[S(Y,V)\eta(X)-S(X,V)\eta(Y)-2ng(Y,V)\eta(X)+2ng(X,V)\eta(Y)].$$

Put $X=\xi$ in above equation, we get

$$S(Y,U)=\Big(\frac{r}{2n}+1\Big)g(Y,V)+\Big(2n+1+\frac{r}{2n}\Big)\eta(Y)\eta(V), \tag{4.4}$$

which shows that '*M*' is an $\eta-$Einstein manifold.

Conversely if the hypersurface *M* is totally umbilical then we have $\lambda=0$. Also,

$$(C(X,Y).H)(U,V)=\lambda[\eta(V).g(C(X,Y)U,\xi)+\eta(U)g(C(X,Y)V,\xi)]. \tag{4.5}$$

Hence, we get $C.H=0$, i.e. *M* is Weyl semi-parallel and if hypersurface *M* is $\eta-$Einstein then by equation (4.4) and (3.22), we get

$$g(C(X,Y)U,\xi)=0, \tag{4.6}$$

from (4.5) and (4.6), we get

$$C.H=0$$

i.e. *M* is Weyl semi-parallel.

Theorem 4.4. Let *M* be a para-Sasakian hypersurface of a Tachibana manifold $\tilde{M}$. Then *M* satisfy $C.H=L_H Q(g,H)$ if and only if either *M* is totally umbilical or $L_H=0$ on *M*.

Proof. Let the hypersurface *M* satisfy

$$C.H=L_H Q(g,H).$$

Using (4.2) and (4.5) in above equation, we get

$$\lambda[\eta(V)g(C(X,Y)U,\xi)+\eta(U).g(c(X,Y).V,\xi)]=L_H.\lambda[-g(Y,U).\eta(X)\eta(V) +g(X,U)\eta(Y)\eta(V)-g(Y,V).\eta(X).\eta(U)+g(X,V)\eta(Y).\eta(U)]. \quad (4.7)$$

Put $V=\xi$ in above equation

$$\lambda[g(C(X,Y)U,\xi]=L_H\lambda[-g(Y,U).\eta(X)+g(X,U)\eta(Y)].$$

Using (3.22) in above and then contracting it with respect to *X*, we get

$$\lambda.S(Y,U)=\lambda\left(\frac{r}{2n}+1+L_H(2n-1)\right)g(Y,U) +\lambda\left(2n+1+\frac{r}{2n}+L_H(2n-1)\right)\eta(Y).\eta(U). \quad (4.8)$$

From (4.8), we get either $\lambda=0$, or

$$S(Y,U)=\left(\frac{r}{2n}+1+L_H(2n-1)\right)g(Y,U) +\left(2n+1+\frac{r}{2n}+L_H(2n-1)\right)\eta(Y).\eta(U). \quad (4.9)$$

If $\lambda=0$, then *M* is totally umbilical with negative unit mean curvature otherwise comparing (4.9) with (4.4), we get

$$L_H=0.$$

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