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PSEUDO RECRRENT MANIFOLDS

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Abstract: A new type of Riemannian manifold has been defined called pseudorecurrent manifold, and some of its geometric properties are derived. Also a non trivial example is obtained to prove the existence.

Keywords: Riemannian manifold, recurrent manifold, Codazzi type Ricci tensor, cyclic Ricci tensor, Ricci symmetric manifold.

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1-Introduction:

It is well known [1] that a non-flat Riemannian manifold is called a recurrent manifold if its curvature tensor R satisfies the relation:

1.1) $(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W$,

where∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A is a non zero 1-form defined as:

1.2) $g(X,\rho) = A(X)$.

The object of this paper is to study a non-flat Riemannian manifold such that its curvature tensor R satisfies the relation:

1.3) $(\nabla_X R)(Y,Z)W = A(X)S(Z,W)Y$,

where∇ and A as stated above and S denote the Ricci tensor such that,

1.4) S(X,Y) = g(LX,Y).

Such a manifold shall be called pseudorecurrent manifold. As in recurrent Riemannian manifold if in particular the 1-form A vanishes identically then the manifold will reduce to symmetric manifold. This will justify the definition (1.3) and the name for it. Also exact justification will be byproducing a concrete example for the manifold as we will see in section 3.

It is known [1] that Bianchi second identity on a Riemannian manifold is as such:

1.5) $(\nabla_X R)(Y, Z, W, U) + (\nabla_W R)(Y, Z, U, X) + (\nabla_U R)(Y, Z, X, W) = 0$

It is also known [1] that on a Riemannian manifold the Ricci tensor is of Codazzi type if,

1.5) $(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = 0,$

and a Riemannian manifold is of cyclic Ricci tensor if,

1.6) $(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(Y, X) = 0.$

In section2 it is shown that every pseudorecurrent manifold is Einstein manifold, and if pseudorecurrent manifold is of Codazzi type Ricci tensor then r and 1 are Eigen value of the Ricci tensor S corresponding to the Eigen vector ρ.But if pseudorecurrent manifold is of cyclic Ricci tensor then $\frac{-r}{2}$ is an Eigen value of the Ricci tensor S and ρ is an Eigen vector corresponding to the Eigen value. Alsoit is shown that on a conformally flat pseudorecurrent manifold the scalar curvature must not be constant.

2-Pseudorecurrent manifold:

Substituting (1.3) on Bianchi second identity we get,

2.1) A(X)S(Z,W)g(Y,U) + A(W)S(Z,U)g(Y,X) + A(U)S(Z,X)g(Y,W) = 0.

Contracting with respect to Y and U we get,

2.2) nA(X)S(X,W) + 2A(W)S(Z,X) = 0.

Again contracting with respect to X and Z yield,

2.3) A(LW) = - 2 A(W).

Contracting (2.2) otherwise we have,

2.4) A(LX) = $\frac{-n^2}{2}$A(W).

Thus we can state,

Theorem2.1) Onpseudorecurrent manifold-2 and $\frac{-n^2}{2}$ are Eigen values of the Ricci tensor S corresponding to the Eigen vectorρ.

Now contracting (1.3) we get,

2.5) $(\nabla_X S)(Y, U) = \mathrm{nA(X)S(Y, U)}.$

It is clear that pseudorecurrent manifold is Ricci symmetric iff it is Ricci flat.

Contracting (1.3) otherwise we get,

2.6) $(\nabla_X S)(Z, W) = \mathrm{rA(X)g(Z, W)},$

where r is the scalar curvature of the manifold.

From (2.5) and (2.6) we can have,

2.3) $S(Z, W) = \frac{r}{n}\mathrm{g(Z, W)}.$

Thus we can state,

Theorem2.2) Every pseudorecurrent manifoldis an Einstein manifold.

If the manifold is of Codazzi type Ricci tensor then by virtue of (1.5) and (2.1) we have,

2.7) $\mathrm{A(X)S(Y, Z) - A(Z)S(Y, X) = 0}.$

Contracting with respect to Y and Z we get,

2.8) $\mathrm{A(LX)} = r\mathrm{A(X)}.$

Thus we can state,

Theorem2.3) Ifpseudorecurrent manifold is of Codazzi type Ricci tensor then r is an Eigen value of the Ricci tensor S and ρ is an Eigen vector corresponding to the Eigen value.
Also by virtue of (1.5) and (2.5) we have,
2.9) $\mathrm{A(X)g(Y,Z) - A(Z)g(Y,X) = 0}$
Contracting with respect to Y and Z we get,
2.10) $\mathrm{A(LX) = A(X)}$.
Thus we can state,
Theorem2.4) If pseudorecurrent manifold is of Codazzi type Ricci tensor then the Ricci tensor S have Eigen value 1 corresponding to the Eigen vectorρ.
Now if the manifold of cyclic Ricci tensor then from (1.6) and (2.5) we have,
2.11) $\mathrm{A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) = 0}$.
Contracting with respect to Y and Z we get,
2.12) $\mathrm{A(LX)} = \frac{-r}{2}\mathrm{A(X)}$.
Thus we can state,
Theorem2.5) Ifpseudorecurrent manifold is of cyclic Ricci tensor then $\frac{-r}{2}$ is an Eigen value of the Ricci tensor S and ρ is an Eigen vector corresponding to the Eigen value.
It is known [1] that in a conformally flat (M^n, g) $(n\geq 3)$,
2.13)$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Z)g(X,Y)]$.
Using (2.6) on this equation we get,
2.14)$\mathrm{r[A(X)g(Y,Z) - A(Z)g(Y,X)]} = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Z)g(X,Y)]$.
Contracting we get,
2.15) $\mathrm{A(X)} = \frac{1}{2\mathrm{r}(\mathrm{n}-1)}dr(X)$.
Thus we can state,
Theorem2.6) On a conformally flat pseudorecurrent manifold the scalar curvature cannot be constant.

3- Example of pseudorecurrent manifold:

Let us consider R^4 endowed with the Riemannian metric [2],

3.1) $d^2 = \mathrm{g_{ij}dx^i dx^j = (1+2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]}$,

(i, j = 1,2,3,4) where $q = \frac{e^{x^1}}{k^2}$ and k is non-zero constant.

Then it is known [2] that the only non vanishing christoffel symbols, Ricci tensors, scalar curvature, curvature tensors, and the covariant derivatives of the curvature tensors are,

3.2) $\Gamma_{22}^1 = \Gamma_{33}^1 = \Gamma_{44}^1 = \frac{q}{1+2q}$; $\Gamma_{11}^1 = \Gamma_{13}^3 = \Gamma_{14}^4 = \frac{q}{1+2q}$,

3.3) $S_{11} = \frac{3q}{(1+2q)^2}$; $S_{22} = S_{33} = S_{44} = \frac{q}{1+2q}$,

3.4) $r = \frac{6q(1+q)}{(1+2q)^3}$,

3.5) $R_{1221} = R_{1331} = R_{1441} = \frac{q}{1+2q}$; $R_{2332} = R_{2442} = R_{4334} = \frac{q^2}{1+2q}$,

3.6) $R_{1221,1} = R_{1331,1} = R_{1441,1} = \frac{q(1-4q)}{(1+2q)^2}$; $R_{2332,1} = R_{2442,1} = R_{4334,1} = \frac{2q^2(1-q)}{(1+2q)^2}$.

Let us define A_i and as follows:

3.7) $$A_i = \begin{cases} \frac{(1-4q)}{(1+2q)^2} & if\ i = j\ in \quad R_{jklm\ ,i} \\ \frac{2q(1-q)}{(1+2q)^2} & if\ i \neq j\ in \quad R_{jklm\ ,i} \end{cases}.$$

To verify the definition by (1.3) we have to verify only the following relations:

3.8) $R_{1221,1} = A_1 S_{22} g_{11}$,

3.9) $R_{1331,1} = A_1 S_{33} g_{11}$,

3.10) $R_{1441,1} = A_1 S_{44} g_{11}$,

3.11) $R_{2332,1} = A_1 S_{33} g_{22}$

3.12) $R_{2442,1} = A_1 A_1 S_{44} g_{22}$,

3.13) $R_{4334,1} = A_1 A_1 S_{33} g_{44}$.

Using (3.1), (3.3) and (3.6) on (3.8) we get,

$$\text{R.H.S.} = A_1 S_{22} g_{11}$$

$$= \frac{(1-4q)}{(1+2q)^2} \left(\frac{q}{1+2q}\right)(1+2q)$$

$$= \frac{q(1-4q)}{(1+2q)^2} = \text{L.H.S.}$$

Similarly we can show (3.9), (3.10). (3.11), (3.12) and (3.13) are true, whereas the other cases are trivially true. Hence R^4 along with the metric g defined by (3.1) is pseudorecurrent manifold. Thus we can state,

Theorem3.1) A Riemannian manifold (M^4,g) endowed with the metric (3.1) is a pseudorecurrent manifold with non constant scalar curvature.

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