

## A Study on Fuzzy Ideal of Near Rings

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### Abstract

#### Keywords:

Near ring;  
Fuzzy set;  
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In this paper we introduced the notion of fuzzy ideals of near rings. A near-ring is a ringoid over the group, i.e. a universal algebra in which an associative multiplication and an addition exist, a near ring is a group with respect to the addition, and the right distributive property must hold too. Zadeh [6] in 1965 introduced the concept fuzzy sets after which several researchers explored on the generalizations of the notion of fuzzy sets and its application to many mathematical branches. A fuzzy set is a class of objects with the continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each of object a grade of membership ranging between zero and one. Abou-Zaid[7], introduced the notion of a fuzzy subnear-ring and studied the fuzzy ideals of a near-ring. Nagarajan [14] introduced the new structures of the Q-fuzzy groups.

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### 1. Introduction

Near-rings were first studied by Fittings in 1932. It is a generalization of a ring. If in a ring we ignore the commutativity of addition and one distributive law then we get a near-ring. G.Pilz [1], J.D.P.Meldrum [2] and many other researchers have contributed and are contributing the near-ring theory. A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In particular, a separation theorem for convex

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fuzzy sets is proved without requiring that the fuzzy sets be disjoint. The fuzzy set was defined with the help of a membership function. We refer usual sets as crisp sets. Every object in the universal set may or may not belong to the crisp set under consideration. But in fuzzy set theory we can say that the object belongs to the crisp set up to certain limit. Generally we express the degree of membership and the degree of truth of any proposition by the real number in  $[0, 1]$ . These sets have a broad utility for expressing the gradual transition from membership to non-membership and conversely. By fuzzy set theory we can express vague concepts into natural language. In 1971 Rosenfeld [3] introduced the concept of fuzzification of groups and defined fuzzy subgroups. Goguen [4] introduced L-Fuzzy Sets. Liu [5] defined the fuzzy ideals of a ring and discussed the operations on fuzzy ideals.

The concept of a fuzzy set was introduced by Zadeh [6] in 1965, utilizing what Rosenfeld [3] defined as fuzzy subgroups. This was studied further in detail by different researchers in various algebraic systems. The concept of a fuzzy ideal of a ring was introduced by Liu [5]. The notion of fuzzy subnear-ring and fuzzy ideals was introduced by Abou-Zaid [7]. Then in many papers, fuzzy ideals of near-rings were discussed. In [8], the idea of fuzzy point and its belongingness to and quasi coincidence with a fuzzy set were used to define  $(\alpha, \beta)$ -fuzzy subgroup, where  $\alpha, \beta$ , take one of the values from  $(\in, q, \in \vee q, \in \wedge q), \alpha \neq \in \wedge q$ . A fuzzy subgroup in the sense of Rosenfeld is in fact an  $(\in, \in)$ -fuzzy subgroup. Thus, the concept of  $(\in, \in \vee q)$ -fuzzy subgroup was introduced and discussed thoroughly in [9]. Bhakat and Das [10] introduced the concept of  $\lambda$ -fuzzy subrings and ideals of a ring. Davvaz [11], Zhan and Davvaz [12] studied a new sort of fuzzy subnear-ring (ideal and prime ideal) called  $(\in, \in \vee q)$ -fuzzy subnear-ring (ideal and prime ideal) and gave characterizations in terms of the level ideals. In [13], the idea of fuzzy ideals of N-groups was defined, and various properties such as fundamental theorem of fuzzy ideals and fuzzy congruence were studied, respectively.

## 2. Preliminaries

A near-ring is defined to be a non-empty set  $R$  with two binary operations “+” and “ $\cdot$ ” satisfying the following axioms:

- (i)  $(R, +)$  is a group,
- (ii)  $(R, \cdot)$  is a semigroup,
- (iii)  $x \cdot (y + z) = x \cdot y + x \cdot z$  for all  $x, y, z \in R$ .

Because of the condition (iii), it is called also a left near-ring. We denote  $xy$  instead of  $x \cdot y$ . Note that  $x0 = 0$  and  $x(-y) = -xy$  but in general  $0x \neq 0$  for some  $x \in R$ . A two sided  $R$ -subgroup of a near-ring  $R$  is a subset  $H$  of  $R$  such that

- (i)  $(H, +)$  is a subgroup of  $(R, +)$ ,
- (ii)  $RH \subset H$ ,
- (iii)  $HR \subset H$ .

If  $H$  satisfies (i) and (ii) then it is called a left  $R$ -subgroup of  $R$ . If  $H$  satisfies (i) and (iii) then it is called a right  $R$ -subgroup of  $R$ . We now review some fuzzy logic concepts. A fuzzy set  $\mu$  in a set  $R$  is a function  $\mu : R \rightarrow [0, 1]$ .

Let  $\text{Im}(\mu)$  denote the image set of  $\mu$ . Let  $\mu$  be a fuzzy set in a set  $R$ . For  $t \in [0, 1]$ , the set  $R_t \mu := \{x \in R \mid \mu(x) \geq t\}$  is called a level subset of  $\mu$ . In what follows the letter  $R$  denotes a near-ring unless otherwise specified. Let  $\mu$  be a fuzzy set in  $R$ . We say that  $\mu$  is a fuzzy subnear-ring of  $R$  if, for all  $x, y \in R$ ,

(F1)  $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ , (F2)  $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ . If a fuzzy set  $\mu$  in  $R$  satisfies the property (F1) then  $\mu(0) \geq \mu(x)$  for all  $x \in R$ . Let  $f$  be a mapping from a set  $R$  to a set  $S$  and let  $\mu$  be a fuzzy set in  $R$ . Then  $f(\mu)$ , the image of  $\mu$ , is a fuzzy set in  $S$  defined by

$$f(\mu)(y) := \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}$$

for all  $y \in S$ .

### 3. Fuzzy ideals in Near Rings

**Definition:** Let  $X$  be a non-empty universal set. A fuzzy subset  $\mu$  of  $X$  is a function  $\mu: X \rightarrow [0,1]$ .

**Definition:** A Fuzzy subset  $\mu$  of a Near-Ring  $N$  is called a Fuzzy sub near-ring of  $N$  if for all  $x, y \in N$

$$\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$$

$$\mu(xy) \geq \min\{\mu(x), \mu(y)\}$$

**Definition:** A Fuzzy subset  $\mu$  of a Near-Ring  $N$  is called a Fuzzy ideal of  $N$  if  $\mu$  is a Fuzzy sub near-ring of  $N$  and

$$\mu(x) = \mu(y + x - y)$$

$$\mu(xy) \geq \mu(y)$$

$$\mu((x + i)y - xy) \geq \mu(i) \text{ for } x, y, i \in N$$

**Definition:** Let  $N$  and  $N'$  be two near rings. Then the mapping  $f: N \rightarrow N'$  is called (near ring) homomorphism if for all  $x, y \in N$ , the following holds

$$(i) f(x + y) = f(x) + f(y) \text{ and}$$

$$(ii) f(xy) = f(x)f(y)$$

**Proposition (1)** If  $A = (\mu_A, \nu_A)$  be IFI in near ring  $N$ , then  $C_\alpha, \beta(A)$  is ideal of  $N$  if  $\mu_A(0) \geq \alpha$ ,  $\nu_A(0) \leq \beta$

**Theorem(1)** If  $A$  and  $B$  be two IFI's of a near ring  $N$ , then  $A \cap B$  is also IFI of  $N$ .

**Proof.** Since  $A$  and  $B$  be two IFI's of near ring  $N$ . By Proposition (1), we have  $C_\alpha, \beta(A)$  and  $C_\alpha, \beta(B)$  are ideals in near ring  $N$ . Since intersection of two ideals in near ring is ideal in  $N$ . Therefore  $C_\alpha, \beta(A) \cap C_\alpha, \beta(B)$  is ideal in  $N \Rightarrow C_\alpha, \beta(A \cap B)$  is ideal in  $N$  (by using Proposition 1)  $\Rightarrow A \cap B$  is IFI in near ring  $N$

**Proposition (2):** Let  $f: X \rightarrow Y$  be a mapping. Then the following holds

$$f(C_{(\alpha, \beta)}(A)) \subseteq C_{(\alpha, \beta)}(f(A)), \forall A \in IFS(X)$$

$$f^{-1}(C_{(\alpha, \beta)}(B)) \subseteq C_{(\alpha, \beta)}(f^{-1}(B)), \forall B \in IFS(Y)$$

**Theorem (2)** If  $A = (\mu_A, \nu_A)$  is an IFS of a near ring  $N$ , then  $A$  is IFI if and only if  $C_\alpha, \beta(A)$  is an ideal of  $N$ , for all  $\alpha, \beta \in [0, 1]$  with  $\alpha + \beta \leq 1$  and  $\mu_A(0) \geq \alpha$ ,  $\nu_A(0) \leq \beta$ .

**Proof** If  $A$  be IFI of a near ring  $N$ , then  $C_\alpha, \beta(A)$  is an ideal of  $N$ , for all  $\alpha, \beta \in [0,1]$  with  $\alpha + \beta \leq 1$  and  $\mu_A(0) \geq \alpha$ ,  $\nu_A(0) \leq \beta$ .

To show that A is IFI of near ring N. Let  $x, y \in N$  and  $\alpha = \min\{\mu_A(x), \mu_A(y)\}$  and  $\beta = \max\{v_A(x), v_A(y)\} \Rightarrow \mu_A(x) \geq \alpha, \mu_A(y) \geq \alpha$  and  $v_A(x) \leq \beta, v_A(y) \leq \beta \Rightarrow \mu_A(x) \geq \alpha, v_A(x) \leq \beta$  and  $\mu_A(y) \geq \alpha, v_A(y) \leq \beta$

Therefore  $x, y \in C_{\alpha, \beta(A)}$ . As  $C_{\alpha, \beta(A)}$  is ideal in near ring N  $\Rightarrow x - y \in C_{\alpha, \beta(A)}$

$$\therefore \mu_A(x - y) \geq \alpha = \min\{\mu_A(x), \mu_A(y)\} \text{ and } v_A(x - y) \leq \beta = \max\{v_A(x), v_A(y)\}$$

Thus  $\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y)\}$  and  $v_A(x - y) \leq \max\{v_A(x), v_A(y)\}$  .....(1) As  $C_{\alpha, \beta(A)}$  N  $\subseteq C_{\alpha, \beta(A)}$  holds for all  $\alpha, \beta \in [0, 1]$  with  $\alpha + \beta \leq 1$  and  $\mu_A(0) \geq \alpha, v_A(0) \leq \beta$ . Let  $x \in C_{\alpha, \beta(A)}$  be s.t  $\mu_A(x) = \alpha$  and  $v_A(x) = \beta$  and  $n \in N$  be any element.

Then  $xn \in C_{\alpha, \beta(A)}$  and so  $\mu_A(xn) \geq \alpha = \mu_A(x)$  and  $v_A(xn) \leq \beta = v_A(x)$  i.e.  $\mu_A(xn) \geq \mu_A(x)$  and  $v_A(xn) \leq v_A(x)$  And if  $x \in C_{\alpha, \beta(A)}$  be such that  $\mu_A(x) = \alpha 1$  and  $v_A(x) = \beta 1 \leq 1 - \alpha 1$ , where  $\alpha 1 \geq \alpha$ .

$\mu_A(xn) \geq \alpha 1 = \mu_A(x)$  and  $v_A(xn) \leq \beta 1 = v_A(x)$  i.e.  $\mu_A(xn) \geq \mu_A(x)$  and  $v_A(xn) \leq v_A(x)$  Thus  $\mu_A(xn) \geq \mu_A(x)$  and  $v_A(xn) \leq v_A(x)$  holds for all  $x, n \in N$  .....(2)

Then  $(y + x - y) \in C_{\alpha, \beta(A)} \Rightarrow \mu_A(y + x - y) \geq \alpha = \mu_A(x)$  and  $v_A(y + x - y) \leq \beta = v_A(x)$  Now, if  $x \in C_{\alpha, \beta(A)}$  be s.t.  $\mu_A(x) = \alpha 1$  and  $v_A(x) = \beta 1 \leq 1 - \alpha 1$ , where  $\alpha 1 \geq \alpha$  Then  $1 1, x() \in C_{\alpha, \beta(A)}$  As  $1 1, C_{\alpha, \beta(A)}$  is normal subgroup of N. So  $1 1, (-y)() yx C_{\alpha, \beta(A)} + \in \alpha \beta \Rightarrow \mu_A(y + x - y) \geq \alpha 1 = \mu_A(x)$  and  $v_A(y + x - y) \leq \beta 1 = v_A(x)$

i.e.  $\mu_A(y + x - y) \geq \mu_A(x)$  and  $v_A(y + x - y) \leq v_A(x)$  holds for all  $x, y \in N$  .....(3)

Next to show that  $\mu_A(n(x + i) - nx) \geq \mu_A(i)$  and  $v_A(n(x + i) - nx) \leq v_A(i)$  holds for all  $x, n \in N, i \in A$ . Take  $i \in C_{\alpha, \beta(A)}$  be an element such that  $\mu_A(i) = \alpha$  and  $v_A(i) = \beta$ . Since  $C_{\alpha, \beta(A)}$  is ideal of the near ring N.

Therefore, for  $x, n \in N$ , we have  $n(x + i) - nx \in C_{\alpha, \beta(A)} \Rightarrow \mu_A(n(x + i) - nx) \geq \alpha = \mu_A(i)$  and  $v_A(n(x + i) - nx) \leq \beta = v_A(i)$  and if  $i \in C_{\alpha, \beta(A)}$  be such that  $\mu_A(i) = \alpha 1$  and  $v_A(i) = \beta 1 \leq 1 - \alpha 1$ , where  $\alpha 1 \geq \alpha$ .

Then  $1 1, i() \in C_{\alpha, \beta(A)}$  As  $1 1, C_{\alpha, \beta(A)}$  is ideal of N.  $\therefore n(x + i) - nx \in 1 1, C_{\alpha, \beta(A)} \Rightarrow \mu_A(n(x + i) - nx) \geq \alpha 1 = \mu_A(i)$  and  $v_A(n(x + i) - nx) \leq \beta 1 = v_A(i)$  i.e.  $\mu_A(n(x + i) - nx) \geq \mu_A(i)$  and  $v_A(n(x + i) - nx) \leq v_A(i)$  .....(4)

From (1), (2), (3) and (4) we find that A is IFI of near ring N.

**Theorem (3)** Let N and N' be two near rings and let  $f: N \rightarrow N'$  be near ring homomorphism. If B =  $(\mu_B, v_B)$  is an IFI in N', then the pre-image  $f^{-1}(B)$  of B under f is an IFI of N.

**Proof.** Since B is IFI in near ring N'  $\Rightarrow C_{\alpha, \beta(B)}$  is ideal in N', for all  $\alpha, \beta \in [0, 1]$  with  $\alpha + \beta \leq 1$  and  $\mu_B(0) \geq \alpha, v_B(0) \leq \beta$  [ By Proposition (1) ]  $\therefore f^{-1}(C_{\alpha, \beta(B)})$  is ideal in N. But  $f^{-1}(C_{\alpha, \beta(B)}) = C_{\alpha, \beta(f^{-1}(B))}$  [By Proposition(2)]  $\Rightarrow C_{\alpha, \beta(f^{-1}(B))}$  is ideal in N and by using Theorem (2), we get  $f^{-1}(B)$  is IFI in near ring N.

**Definition:** Let Q and N denote a set and a near-ring respectively. A mapping  $\mu: N \times Q \rightarrow [0, 1]$  is called a Q-fuzzy set in N.

**Definition :** A Q-fuzzy set  $\mu$  in a near-ring N is called Q-fuzzy sub near-ring of N if

$$(i) \mu(x - y, q) \geq \mu(x, q) \wedge \mu(y, q)$$

$$(ii) \mu(xy, q) \geq \mu(x, q) \wedge \mu(y, q) \text{ for all } x, y \in N \text{ and}$$

$$q \in Q$$

**Theorem(4) :** A Q- fuzzy set  $\mu$  of E is a Q-fuzzy N-subgroup of E if and only if  $\mu_t, t \in [0,1]$  is an N-subgroup E .

**Proof :** Let  $\mu$  be a Q-fuzzy N-subgroup of E.

Let  $x, y \in \mu_t$  and  $q \in Q$  . Then  $\mu(x, q) \geq t, \mu(y, q) \geq t$ . Therefore,  $\mu(x - y, q) \geq \mu(x, q) \wedge \mu(y, q) \geq t$ .

Thus  $x-y \in \mu_t$  .

Also let  $n \in N, q \in Q$ . Then  $x \in \mu_t$  , implies  $\mu(x, q) \geq t$ .

Since  $\mu$  is a Q-fuzzy N-subgroup of E, we have

$$\mu(nx, q) \geq \mu(x, q) \geq t$$

which implies that  $nx \in \mu_t$  . So  $\mu_t$  is an N-subgroup of E.

Thus  $\mu(nx, q) \geq t \geq \mu(x, q)$  .

So  $\mu$  is a Q-fuzzy N-subgroup of E .

**Theorem(5):** Consider an ideal K of N-group E. Then we can have one to one mapping between the set of Q- fuzzy<sup>2</sup> ideals  $\gamma$  of E so that  $\gamma(0, q)$  is equal to  $\gamma(s, q)$  for all “s” in K and the set  $\phi$ , the set of all Q- fuzzy<sup>2</sup> ideal of E/K.

**Proof:** Let  $\gamma$  be Q- fuzzy<sup>2</sup> ideal of E then following theorem3.2 we can show  $\phi(x+K, q) = \sup_{a \in K} \gamma(x+a, q)$  is Q- fuzzy<sup>2</sup> ideal of E/K. Also we have  $\gamma(0, q) = \gamma(s, q)$ .

Now we have  $\gamma(a+s, q) \geq \gamma(a, q)$ . Also  $\gamma(a, q) = \gamma(a+s-s, q) \geq \gamma(a+s, q)$ . Thus we have  $\gamma(a+s, q) = \gamma(a, q)$ , for all  $s \in K$ . Thus  $\phi(a+K, q)$  is equal to  $\gamma(a, q)$ . So the corresponding  $\gamma \mapsto \phi$  is one to one.

Now consider  $\phi$  be Q- fuzzy<sup>2</sup> ideal in E/K. Define Q-fuzzy<sup>2</sup> set  $\gamma$  in E so that  $\gamma(a, q)$  is equal to  $\phi(a+K, q)$ , for all  $a \in K$ .

Let p and m be two element of E and  $n \in N$   $\gamma(p-m, q)$

$$= \phi((p-m)+K, q)$$

$$= \phi((p+K)-(m+K), q) \geq \phi(p+K, q) \wedge \phi(m+K, q)$$

$$= \gamma(p, q) \wedge \gamma(m, q) \gamma(np, q)$$

$$= \phi(np+K, q) \geq \phi(p+K, q)$$

$$= \gamma(p, q).$$

Hence  $\gamma$  is Q- fuzzy<sup>2</sup> subnear-ring in E.

Now let p, m be two element N-group E and n be an element of N.

$$\gamma(p+m, q) = \phi((p+m)+K, q)$$

$$= \phi((p+K)+(m+K), q) \geq \phi(p+K, q) \wedge \phi(m+K, q)$$

$$= \gamma(p, q) \wedge \gamma(m, q) \gamma(p+m-p, q)$$

$$= \phi((p+m-p)+K, q)$$

$$= \phi((p+K)+(m+K)-(p+K), q) \geq \phi(m+K, q)$$

$$= \gamma(m, q) \gamma(n(p+m)-np, q)$$

$$\begin{aligned} &= \phi((n(p+m)-np)+K, q) \\ &= \Phi(n((p+K)+(m+K))-n(p+K), q) \geq \phi(m+K, q) \\ &= \gamma(m, q) \end{aligned}$$

Thus  $\gamma$  is  $Q$ -fuzzy<sup>2</sup> ideal in  $N$ -group  $E$ . Clearly  $\gamma(a, q)$  is equal to  $\phi(a+K, q)$  which is again equal to  $\phi(K, q)$ , for all “ $a$ ” in  $K$ . This indicates that  $\gamma(0, q)$  is equal to  $\gamma(s, q)$  for all “ $s$ ” in  $K$ .

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