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Contributions on Geometric Group Theory

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Abstract: **This Paper Mainly Focus on geometric group theory is to consider finitely generated groups themselves as geometric objects. This is usually done by studying the Cayley graphs of groups, by proving on theorems related to group theory using Metric Spaces.**

Introduction

Geometric group theory is an area in mathematics devoted to the study of finitely generated groups via exploring the connections between algebraic properties of such groups and topological and geometric properties of spaces on which these groups act (that is, when the groups in question are realized as geometric symmetries or continuous transformations of some spaces).

Another important idea in geometric group theory is to consider finitely generated groups themselves as geometric objects. This is usually done by studying the Cayley graphs of groups, which, in addition to the graph structure, are endowed with the structure of a metric space, given by the so-called word metric.

Geometric group theory, as a distinct area, is relatively new, and became a clearly identifiable branch of mathematics in the late 1980s and early 1990s. Geometric group theory closely interacts with low-dimensional topology, hyperbolic geometry, algebraic topology, computational group theory and differential geometry. There are also substantial connections with complexity theory, mathematical logic, the study of Lie Groups and their discrete subgroups, dynamical systems, probability theory, K-theory, and other areas of mathematics.

(CALEY GRAPHS)

Given a group G and A certain kind of multi-subset Γ of G (to be defined shortly), one can construct a graph called a Cayley graph. These graphs are highly symmetric. We can derive properties of the graph from properties of the group, thereby giving us a bridge between graph theory and group theory

***Definition* 1.22:-** let G be a group and Γ be a multi-subset of G. We say that Γ is symmetric if whenever γ is an element of Γ with multiplicity n, then γ^{-1} is an element of Γ of multiplicity *n*. If Γ is symmetric multi-subset of G, then we write $\Gamma \subset G$.

***Example* 1.23:-** Consider the group $\mathbb{Z}_6$. The multi-subset $\Gamma_1 = \{1,1,2,4,5,5\}$ is symmetric. The multi-subset $\Gamma_2 = \{1,5,5\}$ is not symmetric because 5 occurs with multiplicity two, while its inverse 1 occurs with multiplicity one. The multi-subset $\Gamma_3 = \{1,2,4\}$ is not symmetric as the inverse of 1 does not even appear.

***Definition* 1.24:-** Let G be a group and $\Gamma \subset G$. The Cayley graph of G with respect to Γ, denoted by $C(G,\Gamma)$, is defined as follows, The vertices of $C(G,\Gamma)$ are the elements of G. Two vertices $x, y \in G$ are adjacent if and only if there exists $\gamma \in \Gamma$ such that $x = x\gamma$. (In other words, $y^{-1}x \in \Gamma$.) The multiplicity of the edge $\{x,y\}$in the edge multi set E equals the multiplicity of $y^{-1}x$ in Γ.

***Example* 1.25:-** The Cayley graph $C(\mathbb{Z}_4,\{1,1,3,3\})$ is shown in figure 1.5. There are two edges between 1 and 2, for example, because – 1 + 2 = 1 has multiplicity 2 in {1,1,3,3}.

***Remarks* 1.26:-** Why do we want Γ to be symmetric in De. 1.24G suppose that $x = y\gamma$ where $\gamma \in \Gamma$. Then x and y are adjacent. But to make the definition well defined there should exist a $\gamma^1 \in \Gamma$ such that $y = xy^1$. This would imply that $y^1 = y^{-1}$.

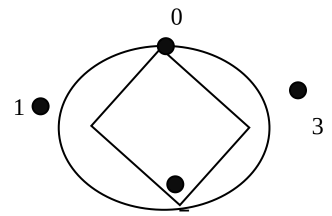
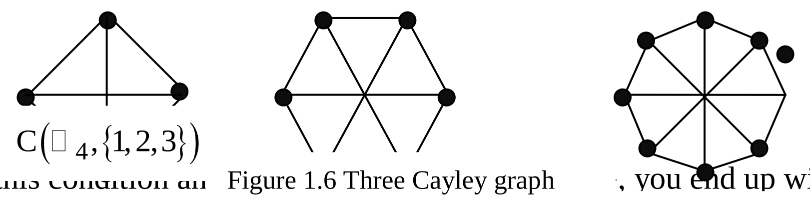


Figure 1.5 $C(\mathbb{Z}_4,\{1,1,3,3\})$

Graph Given values and the isoperimetric constant



$C(\mathbb{Z}_4,\{1,2,3\})$ Figure 1.6 Three Cayley graph $C(\mathbb{Z}_6,\{1,3,5\})$ $C(\mathbb{Z}_8,\{1,4,7\})$

If you relax this condition an..., you end up with a directed Cayley graph. See Def.

***Example* 1.27:-** Figure 1.6 shows the first few graphs in the sequence $\left(C\left(\mathbb{Z}_{2n},\{1,m,2n-1\}\right)\right)$.

***Example* 1.28:-** Recall our conventions regarding the symmetric group S_n. Figure 7.1 shows the Cayley graph $C\left(S_3,\{(1,2),(2,3),(1,2,3),(1,3,2)\}\right)$. You will come to recognize this graph as our "end-of-proof symbol.

Proposition 1.29:- Let G be a group and $\Gamma \subset G$. Then the following are true.:

1. $C(G,\Gamma)$ is $|\Gamma|-$regular

2. $C(G,\Gamma)$ is connected if and only if Γ generates G as a group.

Proof

1. Suppose that $g \in G$ is a vertex of $C(G,\Gamma)$ and that $\Gamma = \{\gamma_1 = ---\gamma_d\}$. Then the neighbors of g are the vertices $g\gamma_1, g\gamma_2, ---g\gamma_d$ (counted with multiplicity). Hence the degree of the vertex g is $d = |\Gamma|$.

2. Let I_G be the identity element of G. Then Γ generates G as group if and only if for every $g \in G$ there exists $\gamma_1 = ---\gamma_k \in \Gamma$ such that $g = \gamma_1 = ---\gamma_k = I_G\gamma_1 ---\gamma_k$. This is equivalent to saying that for every element $g \in G$, there is a walk in the graph X from I_G to g. (The equation $g = \gamma_1 = ---\gamma_k = I_G\gamma_1 ---\gamma_k$ gives the walk $(I_G, I_G\gamma_1, I_G\gamma_1\gamma_2, ---_G\gamma_1\gamma_2 ---\gamma_k)$.) This is equivalent to the fact that X is a connected graph. (For if g, $h \in G$, reverse the walk from g to I_G, then walk from I_G to h.)

Remarks 1.30

Note that if we had counted a loop as contribution 2 to the degree and $I_G \in \Gamma$, then Prop. 1.29 (1) would fail.

2. CLASSICAL EXAMPLES

A classical example of geometric methods used in group theory is the topological proof of Schreier's theorem.

Theorem 2.1 (Schreier's Theorem). *Let G be a free group and $H \subseteq G$ be a sub-group. Then H is free. If the rank $rk(G)$ and the index $[G:H]$ are finite, then the rank of H is finite and satisfies*

$$rk(H) = [G:H].(rk(G)-1)+1.$$

Proof. Let G be a free group on the set S. Take the wedge $X = \vee_s S^1$ of circles, one copy for each element in S. This is a 1-dimensional CW -complex with $\pi_1(X) \cong G$

by the Seifert-van Kampen Theorem. Let $p:\bar{X}\to X$ be the covering associated to $H\subseteq G=\pi_1(X)$ we have $\pi_1(X)\cong H$. Since X is a one dimensional CW-complex, $\bar{X}$ is a one dimensional CW-complex. If $T\subseteq\bar{X}$ is a maximal tree then $\bar{X}$ is homotopy equivalent to $\bar{X}/T=\vee_{\bar{S}}S^{'}$ for some $\bar{S}$. By the Seifert-van Kampen Theorem $H\cong\pi_1(X)$ is the free group generated by $\bar{S}$.

Suppose that $rk(G)$ *and* $[G:H]$are finite. Since $|s|=rK(G)$ the CW -complex X is compact. Since $[G:H]$is finite, the CW -complex $\bar{X}$ and $\bar{X}/T$ are compact. Hence $rk(H)=\bar{S}$ is finite. We obtain for the Euler characteristics.

$$1-|\bar{S}|=\chi(\bar{X})=[G:H].\chi(X)=[G:H].1-|S|$$

Since $|s|=rK(G)$ and $rk(H)=\bar{S}$, the claim follows.

Definition 2.1 Let $X_1=(X_1,d_1)$ and $X_2=(X_2,d_2)$ be two two metric spaces. A map $f:X_1\to X_2$ is called a *quasiisometry* if there exist real numbers $\lambda,\ c>0$ satisfying:

i) The inequality

$$\lambda^{-1}.d_1(x,y)-c\le d_2(f(x),f(y))\le\lambda.d_1(x,y)+c \text{ holds for all } x,y\in X_1;$$

ii) For every x_2 in X_2 there exists x_1 in X_1 with $d_2(f(x_1),x_2)<C$

We call X_1 and X_2 *quasiisometric* if there is a quasiisometry $X_1\to X_2$.

Remark 2.2 (Quasiisometry is an equivalence relation). If $f:X_1\to X_2$ is a quasi-isometry, then there exists a quasiisometry $g:X_1\to X_2$such that both composites $g\circ f$ and $f\circ g$ have bounded distance from the identity map. The composite of two quasiisometries is again a quasiisometry. Hence the notion of quasiisometry is an equivalence relation on the class of metric spaces.

Definition 2.3 (Word-metric). Let G be a finitely generated group. Let Sbe a finite set of generators. The *word metric*

$$d_S:G\times G\to R$$

assigns to (g,h) the minimum over all integers n ≥ 0 such that $g^{-1}h$ can be written as a product $S_1^{\epsilon_1}, S_2^{\epsilon_2}, \cdots, S_n^{\epsilon_n}$ for elements $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$.

The metric d_s depends on S. The main motivation for the notion of quasiisometry is that the quasiisometry class of (G, d_S) is independent of the choice of S by the following elementary lemma.

Lemma 2.4. *Let G be a finitely generated group. Let S_1 and S_2 be two finite sets of generators. Then the identity $id:(G, d_{S_1}) \to (G, d_{S_2})$ is a quasiisometry.*

Proof. Choose λ such that for all $s_1 \in S_1$ we have $d_{S_2}(S_1, e)$, $d_{S_2}(S_1^{-1}, e) \le \lambda$ and for $s_2 \in S_2$ we have $d_{S_1}(S_2, e)$, $d_{S_1}(S_2^{-1}, e) \le \lambda$. Take $C = 0$.

Definition 2.5 (Cayley graph). Let G be a finitely generated group. Consider a finite set S of generators. The *Cayley graph* $\mathrm{Cay}_S(G)$ is the graph whose set of vertices is G and there is an edge joining g_1 and g_2 if and only if $g_1 = g_2 s$ for some $s \in S$.

A *geodesic* in a metric space (X, d) is an isometric embedding $I \to X$, where $I \subset R$ is an interval equipped with the metric induced from the standard metric on R.

Definition 2.6 (Geodesic space). A metric space (X, d) is called a *geodesic space* if for two points $x, y \in X$ there is a geodesic $c:[0, d(x,y)] \to X$ with $c(0) = x$ and $c(d(x,y)) = y$.

Notice that we do not require the unique existence of a geodesic joining two given points.

Remark 2.7 (Metric on the Cayley graph). There is an obvious procedure to define a metric on $Cay_S(G)$ such that each edge is isometric to [0, 1] and such that the distance of two points in $Cay_S(G)$ is the infimum over the length over all piecewise linear paths joining these two points. This metric restricted to G is just the word metric d_S. Obviously the inclusion $(G, d_S) \to Cay_S(G)$ is a quasiisometry. In particular, the quasiisometry class of the metric space $Cay_S(G)$ is independent of S.

Definition 2.10. Two groups G_1 and G_2 are *commensurable* if there are subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that the indices $[G_1 : H_1]$ and $[G_2 : H_2]$ are finite and H_1 and H_2 are isomorphic.

The Cayley graph allows to translate properties of a finitely generated group to properties of a geodesic metric space.

Lemma 2.8 (Švarc-Milnor Lemma). *Let X be a geodesic space. Suppose that G acts properly, cocompactly and isometrically on X. Choose a base point $x \in X$. Then the map*
$f : G \to X,\ g \mapsto gx$
is a quasiisometry.

Example 2.9. Let $M = (M, g)$ be a closed connected Riemannian manifold. Let $\tilde{M}$ be its universal covering. The fundamental group $\pi = \pi_1(M)$ acts freely on $\tilde{M}$. Equip $\tilde{M}$ with unique π-invariant Riemann metric for which the projection with $\tilde{M} \to M$ becomes a local isometry. The fundamental group π is finitely generated.

Equip it with the word metric with respect to any finite set of generators.

Then π and $\tilde{M}$ are quasiisometric by the Svarc-Milnor Lemma 2.8.

Definition 2.10. Two groups G_1 and G_2 are *commensurable* if there are subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that the indices $[G_1 : H_1]$ and $[G_2 : H_2]$ are finite and H_1 and H_2 are isomorphic.

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