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Different Mock Theta functions

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Abstract (10pt)

Ramanujan gave 17 examples of mock theta functions stated numerous identities and asymptotic properties for them. The first detailed description of these functions was given by Watson in his celebrated Presidential Address delivered at the meeting of the London Mathematical Society in November, 1935. In this present communication we list all possible mock theta functions of different orders and give a brief introduction to mock theta functions.

Keywords:

Ramanujan's mock theta functions;
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1. Introduction (10pt)

Some of the famous and extraordinary gifted mathematicians led a tragically short life. Ramanujan is one of the mathematicians among them. Three months before his death, he discovered mock theta functions and he described them in a letter to Hardy which was written under difficulties. Subsequent work stimulated by his letter has been of somewhat different types. The first detailed description of these functions was given by Watson in his celebrated Presidential Address delivered at the meeting of the London Mathematical Society in November, 1935.

Ramanujan's general definition of a mock theta function is a function of $f(q)$ defined by a q-series convergent when $|q| < 1$ which satisfies the following two conditions,

(a) For every root ξ of unity, there exist a θ-function θ(q) such that difference between $f(q)$ and θ(q) is bounded as q $\to \xi$, radially.

(b) There is no single theta function which works for all ξ, i.e. for every θ-function θ(q) there is some root of unity ξ for which $f(q)$ minus the theta function θ(q) is unbounded as q $\to \xi$ radially.

Ramanujan motivated his mock theta functions by first describing briefly two "genuine" theta functions. The first one is the q-series for the partition function p(n) and the second one is the q-

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seies for the partitions into parts of some special forms. According to Ramanujan, a mock theta function is a special kind of q-series that immitates two properties of the above mentioned theta functions. The first property is that they can be expressed in Eulerian form and the second property is that the q-series converge for $|q| < 1$ and have unit circle as the natural boundary.

A study of these mock theta functions and their sums and expansions has been made by Watson [26], Agarwal [1] and Andrews (7). Later on, Andrews and Hickerson (3), Choi (8) and Gordon and Mc Intosh(13) studied certain q-series in the Lost Notebook and named them as sixth, eighth and tenth order mock theta functions.

2. Mock Theta Functions

In this present communication we list all possible mock theta functions of different orders and give a brief introduction to mock theta functions.

Ramanujan gave 17 examples of mock theta functions stated numerous identities and asymptotic properties for them. Andrews found more examples in Ramanujan's "lost" notebook, which he argues was written by Ramanujan during the last phase of his life. Andrews [5], Gordon [15], Selberg [22] and Watson [23] proved Ramanujan's claims and gave transformation formulas connecting them. Andrews and Garvan [2] proved the fifth order mock theta functions which are found in the lost notebook and called them Mock Theta Conjectures. The proof of these and other mock theta functions were given by Hickerson [16], [17].

Andrews [6] listed the following mock theta functions of order 2;

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} \qquad \text{...(2.1)}$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}} \qquad \text{...(2.2)}$$

$$C(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} \qquad \text{...(2.3)}$$

The complete list of mock theta functions of order 3 are

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^2)^2 \ldots\ldots (1+q^n)^2} \qquad \text{...(2.4)}$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)\ldots\ldots(1+q^{2n})} \qquad \text{...(2.5)}$$

$$\psi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3)\ldots\ldots(1-q^{2n-1})} \qquad \text{...(2.6)}$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)(1-q^2+q^4)\ldots\ldots(1-q^n+q^{2n})} \qquad \text{...(2.7)}$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1-q^2)(1-q^3)^2\ldots\ldots(1-q^{2n+1})^2} \qquad \text{...(2.8)}$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1+q)(1+q^3)\ldots\ldots(1+q^{2n+1})} \qquad \text{...(2.9)}$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)(1+q^3+q^6)\ldots\ldots(1+q^{2n+1}+q^{4n+2})} \qquad \text{.................................(2.10)}$$

with $\omega(q)$, $v(q)$ and $\rho(q)$ due to Watson (23, 24), and Dragonette (10). Andrews and Hickerson (4) proved that the series for $\mu(q)$ does not converge, but the series of even and odd partial sums do converge, so $\mu(q)$ is commonly taken as the average of these two values.

Hikami(18) suggests that the following function is related to mock theta functions of order four;

$$D(q) = \sum_{n=0}^{\infty} \frac{q^n(-q^2;q^2)_n}{(q^{n+1};q)_{n+1}} \qquad \text{.................................(2.11)}$$

Mock theta functions of order 5 are;
Ramanujan (21) gave 10 mock theta functions of order five, given by

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n} \qquad \text{.................................(2.12)}$$

$$F_0(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} \qquad \text{.................................(2.13)}$$

$$1+2\psi_0(q) = \sum_{n=0}^{\infty} (-1;q)_n q^{\binom{n+1}{2}} \qquad \text{.................................(2.14)}$$

$$\phi_0(q) = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2} \qquad \text{.................................(2.15)}$$

$$f_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q)_n} \qquad \text{.................................(2.16)}$$

$$F_1(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}} \qquad \text{.................................(2.17)}$$

$$\psi_1(q) = \sum_{n=0}^{\infty} (-q)_n q^{\binom{n+1}{2}} \qquad \text{.................................(2.18)}$$

$$\chi_0(q) = 2F_0(q) - \phi_0(-q) \qquad \text{.................................(2.19)}$$

$$\chi_1(q) = 2F_1(q) + q^{-1}\phi_1(-q) \qquad \text{.................................(2.20)}$$

Note that the notation here follows the standard convention $(-q)_n = (-q;q)_n$.

Mock theta functions of order 6 ;
In Ramanujan's lost notebook VII, Andrews and Hickerson (4) defined the mock theta functions of order 6 as follows;

$$E(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q)_{2n}} \qquad \text{.................................(2.21)}$$

$$F(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q)_{2n+1}} \qquad \text{.................................(2.22)}$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}(-q;q)_n}{(q;q^2)_{n+1}} \qquad \text{.................................(2.23)}$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)(n+2)/2}(-q;q)_n}{(q;q^2)_{n+1}} \qquad \text{.................................(2.24)}$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n(q;q^2)_n}{(-q;q)_n} \qquad \text{.................................(2.25)}$$

$$2\mu(q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n+1}(1+q^n)(q;q^2)_n}{(-q;q)_{n+1}} \qquad \text{.................................(2.26)}$$

$$\gamma(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(q;q)_n}{(q^3;q^3)_n} \qquad \text{.................................(2.27)}$$

Mock theta functions of order 7:
Ramanujan (21, 7) also gave the following three mock theta functions of order seven;

$$G_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^{n+1})_n} \qquad \text{.................................(2.28)}$$

$$G_1(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^n)_n} \qquad \text{.................................(2.29)}$$

$$G_2(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^{n+1})_n} \qquad \text{.................................(2.30)}$$

Mock theta functions of order 8:
Gordon and McIntosh (14) found the following eight mock theta functions of order 8;

$$S_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \qquad \text{.................................(2.31)}$$

$$S_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n} \qquad \text{.................................(2.32)}$$

$$T_0(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \qquad \text{.................................(2.33)}$$

$$T_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \qquad \text{.................................(2.34)}$$

$$U_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} \qquad \text{.................................(2.35)}$$

$$U_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}} \qquad \text{.................................(2.36)}$$

$$V_0(q)=-1+2\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}$$

$$=-1+2\sum_{n=0}^{\infty}\frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}} \qquad \text{.................................(2.37)}$$

$$V_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}} \qquad \text{.................................(2.38)}$$

Mock theta functions of order 10:
Choi (8) gave the following mock theta functions of order 10 (5);

$$H_0(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}} \qquad \text{.....................................(2.39)}$$

$$H_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+1)/2}}{(q;q^2)_{n+1}} \qquad \text{.....................................(2.40)}$$

A number of expositions and survey articles have appeared on mock theta functions and their generalizations; [14], [9], [11], [12], [20], [19], [26]. Duke gave a reasonably short introduction to Ramanujan's mock theta functions which is accessible to a non-specialist. Along with similar lines from Folsom's article but with more details, Duke outlined further developments of Ramanujan's mock theta functions.

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