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A COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS IN FUZZY METRIC SPACE

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Abstract (10pt)

The purpose of this paper is to prove a common fixed point theorem in a fuzzy metric space using the concept of weakly compatible mappings and CLR property.

Keywords:

Fixed point; weakly compatible mappings;

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1. Introduction

The concept of the fuzzy sets was introduced by Zadeh [1]. Kramosil and Michelek have introduced the concept of Fuzzy metric space. Since then many authors proved fixed point theorem using Fuzzy metric space.Grabic [4] obtained the Banach contraction principle in fuzzy version. George and Veeramani [3] have shown that every metric induces a fuzzy metric. Recently Sintunavarat and kuman initiated an interesting property Common Limit in the Range property (CLR property). Aim of this paper to prove a common fixed point for four self maps with the concept of weakly compatible maps and CLR property.

2. Definitions and Preliminaries

Definition 2.1: A binary operation $* : [0,1]\times[0,1] \rightarrow [0,1]$ is called continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a*1=a$ for all $a\in[0,1]$

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(iv) a*b ≤ c*d whenever a ≤c and b≤ d for all a, b,c,d ∈[0,1]

Definition 2.2: A 3-tuple (X, M,*) is said to be fuzzy metric space if X is an arbitrary set,* is continuous t- norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for all x,y,z∈X, s,t>0

(FM-1) M(x,y,0)=0

(FM-2) M(x,y,t)=1 for all t>0 if and only if x=y

(FM-3) M(x,y,t)= M(y,x,t)

(FM-4) M(x,y,t) * M(y,z,s) ≤ M(x,z,t+s)

(FM-5) M(x,y,.) : [0,∞) →[0,1] is left continuous

Example 2.3 (Induced fuzzy metric space): Let (X,d) be a metric space defined a*b=min{a,b} for all x,y∈X and t>0,

$$M(x,y,t) = \frac{t}{t+d(x,y)} \qquad ----(a)$$

Then (X, M,*) is a fuzzy metric space. We call this fuzzy metric *M* induced by metric *d* the standard fuzzy metric. From the above example every metric induces a fuzzy metric but there exist no metric on X satisfying (a).

Definition 2.4: Let (X, M,*) be a fuzzy metric space then a sequence $<x_n>$ in X is said to be convergent to a point x∈X , if

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad for\ all\ t > 0.$$

Definition 2.5: A sequence $<x_n>$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad for\ all\ t > 0\ and\ \ p > 0.$

Definition 2.6: A fuzzy metric space (X, M,*) is said to be complete if every Cauchy sequence is convergent to a point in X.

Lemma 2.7 : For all x,y∈X, M(x,y,.) is non decreasing.

Lemma 2.8: let (X,M,*)be a fuzzy metric space if there exists k∈(0,1) such that M(x,y,kt)≥M(x,y,t) then x=y.

Proposition 2.9: In the fuzzy metric space (X, M,*) if a*a≥a for all a∈[0,1] then a*b= min{a,b}

Definition 2.10: Two self maps S and T of a fuzzy metric space (X,M,*) are said to be compatible mappings if $\lim_{n\to\infty}$ $M(STx_n,TSx_n,t)=1$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z∈X.

Definition 2.11: Two self maps S and T of a fuzzy metric space (X,M,*) are said to be weakly compatible if they commute at their coincidence point. i.e if Su=Tu for some u∈X then STu=TSu.

Definition 2.12:Let A and S be two self maps defined on a metric space (X,d).We say that the mappings A and S satisfy CLR_A property if there exists a sequence $< x_n > \in X$ such that $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} S x_n = A x$.

3. Main Result

3.1 Theorem: Let (X, M,*) be a Fuzzy metric space with a*b=min{a,b} and let A, B, S and T are self maps of X satisfying the following condition conditions

3.1.1 A(X)$\subseteq$T(X) and B(X) $\subseteq$ S(X)

3.1.2 the pairs (A, S) and (B, T) are weakly compatible

3.1.3 the pairs (A,S) or (B,T) satisfy CLR-property

3.1.4 $$[M(Ax,By,kt)]^2 * M(Ax,By,kt)M(Ty,Sx,kt) \geq \{k_1[M(By,Sx,2kt)*M(Ax,Ty,2kt)] + k_2[M(Ax,Sx,kt)*M(By,Ty,kt)]\}M(Ty,Sx,t)$$

where for all x,y in X and $k_1,k_2 \geq 0$,$k_1+k_2 \geq 1$

then the mappings A, B,S and T have a unique common fixed point in X.

Proof :

Assume that the pairs (A,S) satisfy CLR_A-property so there exists a sequence $< x_n > \in$ X such that $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} S x_n = A x$

Since the condition A(X) $\subseteq$T(X) implies there exists a sequence $< y_n > \in$X such that $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} T y_n = A x$

We show that $\lim_{n\to\infty} B y_n = A x$

Put x= x_n and y= y_n in 3. 1.4

$$[M(Ax_n,By_n,kt)]^2 * M(Ax_n,By_n,kt)M(Ty_n,Sx_n,kt) \geq \{k_1[M(By_n,Sx_n,2kt)*M(Ax_n,Ty_n,2kt)] + k_2[M(Ax_n,Sx_n,kt)*M(By_n,Ty_n,kt)]\}M(Ty_n,Sx_n,t)$$

$$[M(Ax,By_n,kt)]^2 * M(Ax,By_n,kt)M(Ax,Ax,kt) \geq \{k_1[M(By_n,Ax,2kt)*M(Ax,Ax,2kt)] + k_2[M(Ax,Ax,kt)*M(By_n,Ax,kt)]\}M(Ax,Ax,t)$$

$$[M(Ax,By_n,kt)]^2 \geq \{k_1[M(By_n,Ax,2kt)] + k_2[M(By_n,Ax,kt)]\}$$

$$[M(Ax,By_n,kt)] \geq \{k_1+k_2\}$$

$$[M(Ax,By_n,kt)] \geq 1$$

implies $\lim_{n\to\infty} B y_n = A x$

now we have $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} S x_n = \lim_{n\to\infty} T y_n = \lim_{n\to\infty} B y_n = A x$

Assume that the condition A(X) $\subseteq$T(X) implies there exists v$\in$X such that Ax=Tv we claim that Bv=Tv

Put x= x_n and y=v in 3.1.4

$$\left[M(Ax_n,Bv,kt)\right]^2 * M(Ax_n,Bv,kt)M(Tv,Sx_n,kt) \geq \{k_1\left[M(Bv,Sx_n,2kt)*M(Ax_n,Tv,2kt)\right] + k_2\left[M(Ax_n,Sx_n,kt)*M(Bv,Tv,kt)\right]\}M(Tv,Sx_n,t)$$

$$\left[M(Ax,Bv,kt)\right]^2 * M(Ax,Bv,kt)M(Tv,Ax,kt) \geq \{k_1\left[M(Bv,Ax,2kt)*M(Ax,Tv,2kt)\right] + k_2\left[M(Ax,Ax,kt)*M(Bv,Tv,kt)\right]\}M(Tv,Ax,t)$$

$$\left[M(Tv,Bv,kt)\right]^2 * M(Tv,Bv,kt)M(Tv,Tv,kt) \geq \{k_1\left[M(Bv,Tv,2kt)*M(Tv,Tv,2kt)\right] + k_2\left[M(Tv,Tv,kt)*M(Bv,Tv,kt)\right]\}M(Tv,Tv,t)$$

$$\left[M(Tv,Bv,kt)\right]^2 \geq \{k_1\left[M(Bv,Tv,2kt)\right] + k_2\left[M(Bv,Tv,kt)\right]\}$$

$$\left[M(Tv,Bv,kt)\right] \geq \{k_1 + k_2\}$$

$$\left[M(Tv,Bv,kt)\right] \geq 1$$

This implies Bv=Tv
Let Ax=w then Bv=Tv=Ax=w

Again since the pair (B,T) is weakly compatible and Bv=Tv=z implies BTv=TBv or Bw=Tw
From the condition B(X) $\subseteq$ S(X) implies there exists u $\in$ X such that Su=Bv=w
To prove Au=w
Put x=u and y=v in 3.1.4

$$\left[M(Au,Bv,kt)\right]^2 * M(Au,Bv,kt)M(Tv,Su,kt) \geq \{k_1\left[M(Bv,Su,2kt)*M(Au,Tv,2kt)\right] + k_2\left[M(Au,Su,kt)*M(Bv,Tv,kt)\right]\}M(Tv,Su,t)$$

$$\left[M(Au,w,kt)\right]^2 * M(Au,w,kt)M(w,w,kt) \geq \{k_1\left[M(w,w,2kt)*M(Au,w,2kt)\right] + k_2\left[M(Au,w,kt)*M(w,w,kt)\right]\}M(w,w,t)$$

$$\left[M(Au,w,kt)\right]^2 \geq \{k_1\left[M(Au,w,2kt)\right] + k_2\left[M(Au,w,kt)\right]\}$$

$$\left[M(Au,w,kt)\right] \geq \{k_1 + k_2\}$$

$$\left[M(Au,w,kt)\right] \geq 1$$

Implies Au=w gives Au=Su=w
Since the pair (A, S) is weakly compatible and Au=Su=w implies ASu=SAu or Aw=Sw.
We show that Aw=w
Put x=w and y=v in 3.1.4

$$[M(Aw,Bv,kt)]^2 * M(Aw,Bv,kt)M(Tv,Sw,kt) \geq \{k_1[M(Bv,Sw,2kt) * M(Aw,Tv,2kt)] + k_2[M(Aw,Sw,kt) * M(Bv,Tv,kt)]\}M(Tv,Sw,t)$$

$$[M(Aw,w,kt)]^2 * M(Aw,w,kt)M(w,Aw,kt) \geq \{k_1[M(w,Aw,2kt) * M(Aw,w,2kt)] + k_2[M(Aw,Aw,kt) * M(w,w,kt)]\}M(w,Aw,t)$$

$$[M(Aw,w,kt)]^2 \geq \{k_1[M(Aw,w,2kt)] + k_2\}M(w,Aw,t)$$

$$[M(Aw,w,kt)] \geq \{k_1[M(Aw,w,kt)] + k_2\}$$

$$[M(Aw,w,kt)] \geq \frac{k_2}{1-k_1}$$

$$[M(Aw,w,kt)] \geq 1$$

Implies Aw=w

To prove Bw=w

Put x=u and y=w

$$[M(Au,Bw,kt)]^2 * M(Au,Bw,kt)M(Tw,Su,kt) \geq \{k_1[M(Bw,Su,2kt) * M(Au,Tw,2kt)] + k_2[M(Au,Su,kt) * M(Bw,Tw,kt)]\}M(Tw,Su,t)$$

$$[M(w,Bw,kt)]^2 * M(w,Bw,kt)M(Bw,w,kt) \geq \{k_1[M(Bw,w,2kt) * M(w,Bw,2kt)] + k_2[M(w,w,kt) * M(Bw,Bw,kt)]\}M(Bw,w,t)$$

$$[M(Bw,w,kt)]^2 \geq \{k_1[M(Bw,w,2kt)] + k_2\}M(Bw,w,t)$$

$$[M(Bw,w,kt)] \geq \{k_1[M(Bw,w,kt)] + k_2\}$$

$$[M(Bw,w,kt)] \geq \frac{k_2}{1-k_1}$$

$$[M(Bw,w,kt)] \geq 1$$

Implies Bw=w

Therefore Aw=Bw=Sw=Tw=w.

Since Aw=Bw=Sw=Tw=w, we get w is a common fixed point of A,B,S and T.

Uniqueness: Let z(≠w) be the common fixed point of A, B, S and T then we get Az=Bz=Sz=Tz=z.

Put x=w and y=z in 3.1.4

$$[M(Aw,Bz,kt)]^2 * M(Aw,Bz,kt)M(Tz,Sw,kt) \geq \{k_1[M(Bz,Sw,2kt) * M(Aw,Tz,2kt)] + k_2[M(Aw,Sw,kt) * M(Bz,Tz,kt)]\}M(Tz,Sw,t)$$

$$[M(w,z,kt)]^2 * M(w,z,kt)M(z,w,kt) \geq \{k_1[M(z,w,2kt) * M(w,z,2kt)] + k_2[M(w,w,kt) * M(z,z,kt)]\}M(z,w,t)$$

$$[M(w,z,kt)]^2 \geq \{k_1[M(z,w,2kt)] + k_2\}M(z,w,t)$$

$$[M(w,z,kt)] \geq \{k_1[M(z,w,2kt)] + k_2\}$$

$$[M(w,z,kt)] \geq \frac{k_2}{1-k_1}$$

$$[M(w,z,kt)] \geq 1$$

implies w=z

which gives Self maps A,B,S and T have unique common fixed point

3.2 Example: Let X= [0, 1), $M(x,y,t)=\dfrac{t}{t+d(x,y)}$ where d(x,y)=$|x-y|$

$$Ax=Bx=\begin{cases}\frac{1}{4} & if\ 0\le x<\frac{1}{2}\\ \frac{1}{2} & if\ \frac{1}{2}\le x<1\end{cases}\qquad Sx=Tx=\begin{cases}\frac{1}{6} & if\ 0\le x<\frac{1}{2}\\ 1-x & if\ \frac{1}{2}\le x<1\end{cases}$$

then A(X) =B(X)= $\left\{\frac{1}{4},\frac{1}{2}\right\}$ while S(X) =T(X) = $\left\{\frac{1}{6}\cup\left[0,\frac{1}{2}\right]\right\}$ so that the conditions A(X) $\subseteq$T(X) and B(X) $\subseteq$S(X) are satisfied .

Clearly the pairs (A,S) and (B,T) are weakly compatible as they commute at coincident point 1/2 .

Let a sequence x_n= $\left(\frac{1}{2}+\frac{1}{n}\right)$ for n≥1, then $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} S x_n = \frac{1}{2}$ and $A\left(\frac{1}{2}\right)=\frac{1}{2}$ which implies the pair (A,S) satisfies CLR_A property.

Also note that none of the mappings are continuous and the rational inequality holds for the values of $0\le k_1+k_2\ge 1,\ where\ \ k_1,k_2\ge 0$.Therefore all the conditions of Theorem 3.1 are satisfied. Clearly 1/2 is the unique common fixed point of A, B, S and T.

4 Conclusion: Theorem 3.1 proved without using complete fuzzy metric space and without continuous mappings.

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