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Fixed Point Theorems in Metric Spaces and its Applications to Cone Metric Spaces

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Abstract

Fixed point theorems provide conditions under which maps (single or multi-valued) have solutions. The theory itself is a beautiful mixture of analysis, topology, and geometry. In particular, fixed point techniques have been applied in such diverse fields as Biology, Chemistry, Economics, Engineering, Game Theory, and Physics. Fixed point theory plays an important role in functional analysis, approximation theory, differential equations and applications such as boundary value problems etc. The concept of a metric space was introduced in 1906 by M. Frechet [2]. One extension of metric spaces is the so called cone metric space. In cone metric spaces, the metric is no longer a positive number but a vector, in general an element of a Banach space equipped with a cone. In 2007, Huang & Xian [3] introduced the notion of a cone metric space and established some fixed point theorems in cone metric spaces, an ambient space which is obtained by replacing the real axis in the definition of the distance, by an ordered real Banach space whose order is induced by a normal cone P. In this paper we introduce the notion of quasi generalized contraction pair of self maps on a cone metric space and observe that this notion is weaker than the notion of generalized contraction pair of self maps introduced in [1]. Also we prove a common fixed point theorem for a quasi generalized contraction pair of self maps and provide examples analyzing various situations.

Keywords:

Complete metric space,
cone metric space,
contraction pair,
fixed point,
generalized contraction pair,
Quasi generalized contraction pair.

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1. Introduction

One extension of metric spaces is the so called cone metric space. In cone metric spaces, the metric is no longer a positive number but a vector, in general an element of a Banach space equipped with a cone. In 2007, Huang & Xian [3] introduced the notion of a cone metric space and established some fixed point theorems in cone metric spaces, an ambient space which is obtained by replacing the real axis in the definition of the distance, by an ordered real Banach space whose order is induced by a normal cone P. These authors also proved some fixed point theorems of contractive mappings on complete cone metric spaces.

Latter, in 2008 Rezapore and Hamlbarani [4] proved some of the results of Guang and Xiang [3] by omitting the assumption of normality on the cone.

G.V.R. Babu, G.N. Alameyehu and K.N.V.V. Varaprasad [1] proved common fixed point theorems for generalized contraction pair of self maps on a complete cone metric space. In this paper we introduce the notion of quasi generalized contraction pair of self maps and observe that every quasi generalized contraction pair of self maps is a generalized contraction pair. We show that a quasi generalized contraction pair need not be a generalized contraction pair and prove some theorems and provide examples in support of our results.

1.1 Definition:

A metric on a non empty set X is a function (called the distance function or simply distance) d : X × X → R (where R is the set of real numbers). For all *x*, *y*, *z* inX, this function is required to satisfy the following conditions:

1. d(*x*, *y*) ≥ 0 (non-negativity)
2. d(*x*, *y*) = 0 if and only if *x* = *y*
3. d(*x*, *y*) = d(*y*, *x*) (symmetry)
4. d(*x*, *z*) ≤ d(*x*, *y*) + d(*y*, *z*) (subadditivity / triangle inequality).

The first condition is implied by the others.

1.2 Definition:

A semi metric on X is a function d :X × X → R that satisfies the first three axioms, but not necessarily the triangle inequality:

1. d(*x*, *y*) ≥ 0
2. d(*x*, *y*) = 0 if and only if *x* = *y*
3. d(*x*, *y*) = d(*y*, *x*)

1.3 Definition: (Huang & Xian [3]) Let E be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, non-empty and P ≠ { 0 }

(ii) $ax + by \in P \ \forall\ x, y \in P$ and non-negative real numbers *a* and b.

(iii) $P \cap (-P) = \{0\}$.

1.4 Definition: (L. G. Huang, Z. Xian ,[3])

We define a partial ordering $\leq$ on E with respect to P and $P \subset E$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x << y$ if $y - x \in \text{int}\,P$, $\text{int}\,P$ denotes the interior of P. We denote by $\|.\|$ the norm on E. The cone P is called normal if there is a number K > 0 such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$||x|| \leq K\,||y||$ … (1.4.1)

The least positive number K satisfying (1.4.1) is called the normal constant of P.

1.5 Definition: (L. G. Huang, Z. Xian ,[3])

A cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$,then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$

1.6 Definition: (L. G. Huang, Z. Xian ,[3])

E is a real Banach space, P is a cone in E with $\text{int}P \neq \emptyset$ and $\leq$ is the partial ordering with respect to P. Let X be a non-empty set and $d: X \times X \to P$ a mapping such that.

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ (non - negativity)

(d_2) $d(x, y) = 0$ if and only if $x = y$.

(d_3) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)

(d_4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality)

Then d is called a cone-metric on X and (X, d) is called a cone metric space.

1.7 Example: (L. G. Huang, Z. Xian ,[3])

Let $E = R^2, P = \{(x, y) \in E / x, y \geq 0\}, X = R$ and $d : X \times X \to P$ defined by

$d(x, y) = (|x - y|, \alpha|x - y|)$ where $\alpha \geq 0$ is a constant.
Then (X , d) be a cone metric space.

2.Definition: (L. G. Huang, Z. Xian ,[3])

Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$.

We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

(ii) $\{x_n\}_{n\geq 1}$ is said to be a Cauchy Sequence if for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x_m) << c$ for all $n, m \geq N$.

(iii) (X, d) is called a complete cone metric space if every Cauchy Sequence in X is convergent.

In 2008, Rezapour and Hamlbarani [4] , proved that there are no normal cones with normal constant M < 1. Further, in [4] it is shown that for k> 1 there are cones with normal constant M > k. An example of a non normal cone is given in [4]. Further, Rezapour andHamlbarani [4] obtained generalizations of the results of L. G. Huang, Z. Xian [3] by removing the assumption of normality of the cone.

2.1 Lemma : (Rezapour and Hamlbarani[4] , Lemma 1.1)
Every regular cone is normal.

2.2 Lemma : (Rezapour and Hamlbarani [4] , Lemma 2.1)
There is no normal cone with normal constant $M < 1$.
The following examples of cone metric spaces are given in [4].

2.3 Example: (Rezapour and Hamlbarani [4] , Example 2.1)
Let $E = C_R([0,1])$, $C_R([0,1]) = \{ f / f : [0,1] \to R$ is continuous $\}$ endowed with the supremum norm and $P = \{ f \in E / f \geq 0 \}$. Then P is a cone with normal constant M = 1

2.4 Example: (Rezapour and Hamlbarani[4] , Example 2.2)
Let $E = l^1$, $P = \{\{x_n\} \in E / x_n \geq 0$ for all $n\}$, (X, ρ) be a metric space and $d: X \times X \to P$ be defined by $d(x, y) = \{\rho(x, y) / 2^n\}_{n \geq 1}$. Then (X , d) is a cone metric space and the normal constant of P is 1.

2.5 Proposition 0.28 : (Rezapour and Hamlbarani[4] , Proposition 2.2)
For each k >1 , there is a normal cone with normal constant M > k

The following is an example of a non normal cone.

2.6 Example : (Rezapour and Hamlbarani[4] , Example 2.3)
Let $E = C_R^2([0,1])$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, and consider the cone
$P = \{ f \in E : f \geq 0 \}$ .For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$.
Then $0 \leq g \leq f$, $\|f\| = 2$ and $\|g\| = 2k+1$. Since $k\|f\| < \|g\|$, k is not normal constant of P. Therefore , P is a non normal cone.

2.7 Definition : (G.V.R.Babu, G.N.Alameyehu and K.N.V.V.Varaprasad [1])

Let (X, d) be a cone metric space and P be a cone with non empty interior. Let f, g be self maps on X. Suppose that there exists a constant $k \in (0,1)$ and
$p(x,y) \in \{ d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2} \}$ such that
$d(fx, gy) \leq k\, p(x,y)$
for all x, y in X. Then the pair *(f, g)* is called a generalized contraction pair on X.

G.V.R. Babu, G.N. Alameyehu and K.N.V.V. Varaprasad [1] proved the following fixed point theorem in a complete cone metric space, for a generalized contraction pair on X.

2.7 Theorem : ([1], Theorem 2.1) Let *(X, d)* be a complete cone metric space. Suppose that (*h, k*) is a generalized contraction pair on *X* .Then *h* and *k* have a unique common fixed point in *X.*

3. Main results

In this section we introduce the notion of a quasi generalized contraction pair of self maps on a cone metric space (X, *d)*. We observe that every generalized contraction pair is a quasi generalized contraction pair and the other way is not true.

3.1 Definition : Let *(X, d)* be a cone metric space and (*h, k)* be a pair of self maps on *X.* The pair (*h , k)* is said to be a quasi generalized contraction pair on X if there exists $\mu \in (0,1)$ such that the following conditions hold.

$(i)\ d(hx, khx) \le \mu\ d(x, hx)\ and$

$(ii) d(kx, hkx) \le \ \mu\, d(x, kx)\ for\ all\ x \in X.$

We observe that every generalized contraction pair is a quasi generalized contraction pair.

We have the following lemma.

3.2 Lemma : For any quasi generalized contraction pair (*h, k)* of self maps on a cone metric space X ,the fixed point sets of *h* and *k* are the same .

Proof : Follows from the definition of a quasi generalized contraction pair.

3.3 Remark : In Lemma 2.2 , if (h, k) *is a* generalized contraction pair, then the fixed point set of *h* (and hence of *k)* is at most singleton.

The following example shows that a quasi generalized contraction pair need not be a generalized contraction pair.

3.4 Example : Let $X = \left\{\frac{1}{(n+1)^2} : n = 1, 2, \ldots\right\} \cup \{0\}$ and *d* be the usual metric on X. Define $h\colon X \to X$ by $h(x) = x^2 if\ x\ \in\ X\ and\ x\ \neq 0, h(0) = \frac{1}{4}\ and\ put\ k = h.$

Then the pair (*h , k) is a*quasi generalized contraction pair but not a generalized contraction pair, because this pair has no common fixed points in view of Theorem 1.16.

3.5 Notation: Suppose *h* and *k* are self maps on a non empty set X and $x_0 \in X$. Then we define sequence $\{\ x_n\}$ in *X* iteratively as follows:$x_1 = h(x_0),\ x_2 = k(x_1)$ and

In general $x_{2n+1} = h(x_{2n})\ and\ x_{2n+2} = k(x_{2n+1})$, for $n = 0, 1, 2 \ldots$ ... (2.5.1)

3.6 Theorem : Let Xbe a complete cone metric space and (h, k) be a quasi generalized contraction pair on X. Then we have the following

(i) The sequence defined in (2.5.1) is a Cauchy sequence in X and hence is convergent in X

(ii) If one of the two functions *h* and is continuous then the fixed point sets of *h* and *k* arenon empty.

Proof : Let us observe that for any $n \in N$,

$(a)\ d(x_{2n}, x_{2n+1})\ \le\ \mu^{2n} d(x_0, x_1)\ and$

$(b) d(x_{2n+1}, x_{2n+2}) \le\ \mu^{2n+1} d(x_0, x_1)\ for\ the\ sequence \{x_n\}$denoted by (2.5.1) in X and $\mu \in (0,1)$ satifying (i) and (ii) of Definition 2.1. To see this,

we have that $d(x_{2n}, x_{2n+1})\ =\ d(kx_{2n-1}, hx_{2n})$

(by the definition of the sequence $\{x_n\}$)

$= d(kx_{2n-1}, hkx_{2n-1})$

$\le \mu\, d(x_{2n-1}, kx_{2n-1})$

$= \mu\, d(hx_{2n-2}, kx_{2n-2}) \le \mu^2\, d(x_{2n-2}, x_{2n-1})$

Continuing this way we get (a) after a finite number steps.

In a similar way (b) can be established.

Now, from (a) and (b) it follows that

$d(x_n, x_{n+1})\ \le \mu^n d(x_0, x_1)$for any $n \ge 1$ ... (2.6.1)

Let n, m be positive integers such that $n\ <\ m$.

Then $d(x_n, x_m)\ \le\ d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m)$

$\le \mu^n d(x_0, x_1) + \mu^{n+1} d(x_0, x_1) + \cdots + \mu^{m-1} d(x_0, x_1)$ (by 2.6.1)

$= (\mu^n + \mu^{n+1} + \cdots + \mu^{m-1}) d(x_0, x_1)$

$\leq \{\frac{\mu^n}{1-\mu}\}d(x_0,x_1) \to 0 \; as \; n \to \infty$,since$0 < \mu < 1$.

Hence the sequence $\{x_n\}$ is Cauchy. Since the space X is complete, this sequence $\{x_n\}$ converges to some point *x* in X.

Now suppose *h*is continuous.

Then, we have that $h(x) = h(lim x_{2n}) = \lim h(x_{2n}) = lim \, x_{2n+1} = x$.

Thus *x* is a fixed point of *h* and hence the fixed point set of *h* isnon empty. Now by Lemma 2.2 the fixed point set of k is also non empty. Similarly we can prove thatif k is continuous then *x* is fixed point of k. Consequently the fixed point sets of *h*and k are non empty.

3.7 Remark : If $h = k$ = identity function on Xthen the pair *(h, k) is*quasi generalized contraction pair. Further the fixed point sets of *h*and k are the same and may contain more than one fixed point.

3.8 Remark : If both *h* and $k$are not continuous in theorem 2.6, then the fixed point sets of *h* and k may be empty.

This is justified in the following example:

3.9 Example : Let $X = [0,\frac{1}{2}]$ and *d* usual metric on X.

Define $h: X \to X$ by $h(x) = x^2 \; if \; x \in (0,\frac{1}{2}]$ and $h(0) = \frac{1}{2}$ and put $k = h$.

Then the pair (*h*, , k) is a quasi generalized contraction pair. Also both *h* and k are not continuous and *h* and k have no fixed points inX.

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