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Bernoulli Wavelet based Numerical Method for the Solution of Abel's Integral Equations

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Abstract

Bernoulli wavelet based numerical method is developed for the solution of Abel's integral equations. The properties of Bernoulli wavelets are discussed. This method is based on Bernoulli wavelets polynomials. Integral equation is reduced into system of algebraic equations. Using Matlab, we obtained the Bernoulli wavelet coefficients at the collocation points. Numerical results and error analysis are tested through some of the illustrative examples, which show the efficiency of the proposed method.

Keywords:

Abel's integral equations; Bernoulli wavelet; Bernoulli polynomials.

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1. Introduction

Abel's integral equation finds its applications in various fields of science and engineering. Such as microscopy, seismology, semiconductors, scattering theory, heat conduction, metallurgy, fluid flow, chemical reactions, plasma diagnostics, *X*-ray radiography, physical electronics, nuclear physics [18, 19, 8].
In 1823, Abel, when generalizing the tautochrone problem derived the equation:

$$\int_0^t \frac{y(s)}{\sqrt{t-s}}\,ds = f(t), \tag{1.1}$$

where $f(t)$ is a known function and $y(t)$ is an unknown function to be determined. This equation is a particular case of a linear Volterra integral equation of the first kind. For solving Eq. (1.1) different numerical based methods have been developed over the fast few years, such as product integration methods [1, 2], collocation methods [3], homotopy analysis transform method [14]. The generalized Abel's integral equations on a finite segment appeared for the first time in the paper of Zeilon [22]. There are several numerical methods for approximating the solution of singular integral equations is known. Baker [1] studied the numerical treatment of integral equations. A numerical solution of weakly singular Volterra integral equations was introduced in [4]. Babolian and Salimi [5] discussed an operational matrix method based on block-pulse functions for singular integral equations.

Wavelets theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation.

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Wavelets permit the accurate representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [6, 7]. Since from 1991 the various types of wavelet method have been applied for numerical solution of different kinds of integral equation, a detailed survey on these papers can be found in [9]. Such as Lepik et al. [9] applied the Haar wavelets. Maleknejad et al. proposed Legendre wavelets [10], Rationalized haar wavelet [11], Hermite Cubic splines [12], Coifman wavelet as scaling functions [13]. Yousefi et al. [20] have introduced a new CAS wavelet. Shiralashetti and Mundewadi [15] applied the Bernoulli wavelet for the numerical solution of Fredholm integral equations. Some of the papers are found for solving Abel's integral equations using the wavelet based methods, such as Legendre wavelets [21] and Chebyshev wavelets [16]. In this paper, we introduced the numerical method based on Bernoulli wavelets method for solving Abel's integral equations of second kind.

The article is organized as follows: In Section 2, the basic formulation of Bernoulli wavelets and the function approximation is presented. Section 3 is devoted the method of solution. In section 4, numerical results are demonstrated the accuracy of the proposed method using some of the illustrative examples. Lastly, the conclusion of the proposed method is given in section 5.

2. Bernoulli Wavelets and Function Approximation

Bernoulli wavelets are $B_{n,m} = B(k,\hat{n},m,t)$ have four arguments; $\hat{n} = n-1,\ n = 1,2,3,\dots,2^{k-1}$, k is any positive integer, m is the order of Bernoulli polynomials and t is the normalized time. Then it can be defined on the interval [0, 1) as follows,

$$B_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}}\tilde{\beta}_m(2^{k-1}t-\hat{n}), & \frac{\hat{n}}{2^{k-1}} \le t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & otherwise \end{cases} \tag{2.1}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1, & m = 0, \\ \dfrac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}\beta_m(t), & m > 0, \end{cases}$$

where m = 0,1,2,...,M-1 and n = 1,2,...,2^{k-1}. The coefficient $\frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}}$ is for normality, $2^{-(k-1)}$is the dilation parameter, $\hat{n}2^{-(k-1)}$is the translation parameter and

$$\beta_m(t) = \sum_{i=0}^{m}\binom{m}{i}\alpha_{m-i}t^{i}$$

are the well-known Bernoulli polynomials of order m. Where $\alpha_i, i = 0,1,\dots,m$ are Bernoulli numbers. These numbers are a sequence of signed rational numbers which arise in the series expansion of trigonometric functions and can be defined by the identity,

$$\frac{t}{e^t-1} = \sum_{i=0}^{\infty}\alpha_i\frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1, \alpha_1 = \frac{-1}{2}, \alpha_2 = \frac{1}{6}, \alpha_4 = \frac{-1}{30}, \alpha_6 = \frac{1}{42}, \alpha_8 = \frac{-1}{30}, \alpha_{10} = \frac{5}{66}, \ \dots$$

With $\alpha_{2i+1} = 0, i = 1,2,3,\dots$

The first few Bernoulli Polynomials are,

$\beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6},$

$\beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \quad \beta_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30},$

$\beta_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t, \ \beta_6 = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t^2 + \frac{1}{42}, \dots$

A function $f(t) \in L^2[0,1]$ may be expanded as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}B_{n,m}(t), \tag{2.2}$$

where

$$c_{n,m} = (f(t), B_{n,m}(t)). \tag{2.3}$$

In (2.3), (. , .) denotes the inner product.
If the infinite series in (2.2) is truncated, then (2.2) can be rewritten as:

$$f(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}B_{n,m}(t)=C^T\Psi(t), \tag{2.4}$$

where C and $B(t)$ are $2^{k-1}M\times 1$ matrices given by:

$$C=[c_{10},c_{11},...,c_{1,M-1},c_{20},...,c_{2,M-1},...,c_{2^{k-1},0},...,c_{2^{k-1},M-1}]^T$$
$$=[c_1,c_2,...,c_{2^{k-1}M}]^T, \tag{2.5}$$

and

$$\Psi(t)=[B_{10}(t),B_{11}(t),...,B_{1,M-1}(t),B_{20}(t),...,B_{2,M-1}(t),...,B_{2^{k-1},0}(t),...,B_{2^{k-1},M-1}(t)]^T$$
$$=[B_1(t),B_2(t),...,B_{2^{k-1}M}(t)]^T. \tag{2.6}$$

3. Method of Solution

Consider the Abel integral equation,

$$\text{First kind: } f(t)=\int_0^t\frac{y(s)}{\sqrt{t-s}}ds, \quad 0\le t\le 1, \tag{3.1}$$

$$\text{Second kind: } y(t)=f(t)+\int_0^t\frac{y(s)}{\sqrt{t-s}}ds, \quad 0\le t\le 1 \tag{3.2}$$

Numerical procedure as follows:
STEP 1: We first approximate $y(t)$ as truncated series defined in Eq. (2.4). That is,

$$u(t)=Y^T\Psi(t) \tag{3.3}$$

where Y and $\Psi(t)$ are defined similarly to Eqs. (2.5) and (2.6).
STEP 2: Then substituting Eq. (3.3) in Eqs. (3.1) and (3.2), we get

$$\text{First kind: } f(t)=\int_0^t\frac{Y^T\Psi(s)}{(t-s)^\alpha}ds, \tag{3.4}$$

$$\text{Second kind: } Y^T\Psi(t)=f(t)+\int_0^t\frac{Y^T\Psi(s)}{\sqrt{t-s}}ds, \quad 0\le t\ s\le 1 \tag{3.5}$$

STEP 3: Substituting the collocation point t_i in Eqs. (3.4) and (3.5), we obtain,

$$\text{First kind: } f(t_i)=\int_0^{t_i}\frac{Y^T\Psi(s)}{(t_i-s)^\alpha}ds \tag{3.6}$$

$$f(t_i)=Y^TG_1, \text{ where } G_1=\int_0^{t_i}\frac{Y^T\Psi(s)}{(t_i-s)^\alpha}ds$$

$$\text{Second kind: } Y^T\Psi(t_i)=f(t_i)+\int_0^{t_i}\frac{Y^T\Psi(s)}{\sqrt{t_i-s}}ds, \tag{3.7}$$

$$Y^T(\Psi(t_i)-G_2)=f, \text{ where } G_2=\int_0^{t_i}\frac{Y^T\Psi(s)}{\sqrt{t_i-s}}ds$$

STEP 4: Now, we get the system of algebraic equations with unknown coefficients.

First kind: $f=Y^TG_1$

Second kind: $Y^TK=f$, where $K=(\Psi(t_i)-G_2)$

STEP 5: By solving the above system of equations, we obtain the Bernoulli wavelet coefficients 'Y' and then substitute in Eq. (3.3), we obtain the approximate solution of Eq. (3.1) and Eq. (3.2).

4. Numerical experiments

In this section, we present Bernoulli wavelet method for the numerical solution of Abel's integral equations in comparison with existing method [16] to demonstrate the capability of the present method and error analysis are shown in tables and figures. Error function is presented to verify the accuracy and efficiency of the following numerical results:

$$E_{Max} = Error\ function = \|y_e(t_i) - y_a(t_i)\|_2 = \sqrt{\sum_{i=1}^{n} (y_e(t_i) - y_a(t_i))^2}$$

where, y_e and y_a are the exact and approximate solution respectively.

Example 1. Consider the Abel's integral equation of the second kind [16],

$$4y(t) = \frac{4}{\sqrt{t+1}} - \arcsin\left(\frac{1-t}{1+t}\right) + \frac{\pi}{2} - \int_0^t \frac{y(s)}{\sqrt{t-s}} ds, \quad 0 \le t < 1. \tag{4.1}$$

which has the exact solution $y(t) = \dfrac{1}{\sqrt{t+1}}$. Applying the Bernoulli wavelet method for solving Eq. (4.1) with k = 1 and M = 6, we find,

$$f = [4.4051 \quad 4.5050 \quad 4.5071 \quad 4.4834 \quad 4.4512 \quad 4.4166]$$

$$K = \begin{bmatrix} 4.5774 & 5.0000 & 5.2910 & 5.5275 & 5.7321 & 5.9149 \\ -6.6624 & -4.6188 & -2.1485 & 0.5668 & 3.4641 & 6.5105 \\ 5.7342 & -0.6708 & -4.4202 & -4.9344 & -1.8926 & 4.9400 \\ 4.0662 & 6.6074 & 3.5855 & -1.7742 & -5.7928 & -4.4974 \\ -28.5997 & -1.2138 & 24.6731 & 27.8210 & 5.3403 & -25.7715 \\ -3.3038 & -6.7559 & -3.9711 & 2.1422 & 6.0648 & 3.7624 \end{bmatrix}$$

Next, we get the Bernoulli wavelet coefficients,

$$Y = [0.8284 \quad -0.0845 \quad 0.0119 \quad -0.0034 \quad 0.0002 \quad -0.0005]$$

and substituting these coefficients in Eq. (3.3), we get the approximate solution of Eq. (4.1) with exact solution as shown in table 1 and the error analysis is shown in table 2.

Table 1: Numerical result of the example 4.1.

t	Exact	Bernoulli Wavelet (k = 1, M = 6)	Absolute Error
0.0833	0.9608	0.9608	2.43e-06
0.2500	0.8944	0.8944	3.77e-07
0.4167	0.8402	0.8402	5.68e-07
0.5833	0.7947	0.7947	2.71e-07
0.7500	0.7559	0.7559	4.44e-07
0.9167	0.7223	0.7223	2.55e-07

Table 2: Maximum error analysis of the example 4.1

$N = 2^{k-1}M$	Bernoulli Wavelet
k = 1, M = 3	3.15e-04
k = 1, M = 5	1.17e-05
k = 1, M = 6	2.43e-06

Example 2. Next, consider [16],

$$y(t) = 2\sqrt{t} - \int_0^t \frac{y(s)}{\sqrt{t-s}} ds, \quad 0 \le t < 1. \tag{4.2}$$

which has the exact solution $y(t) = 1 - \exp(\pi t)\, erfc\left(\sqrt{\pi t}\right)$. We solved the Eq. (4.2) by approaching the present method for k = 1 and M = 6 is given as,

$$f = [0.5774 \quad 1.0000 \quad 1.2910 \quad 1.5275 \quad 1.7321 \quad 1.9149]$$

$$K = \begin{bmatrix} 1.5774 & 2.0000 & 2.2910 & 2.5275 & 2.7321 & 2.9149 \\ -2.3323 & -2.0207 & -1.2825 & -0.2993 & 0.8660 & 2.1804 \\ 2.1006 & 0.1677 & -1.3456 & -1.8598 & -1.0541 & 1.3064 \\ 1.2988 & 2.5317 & 1.8244 & -0.0131 & -1.7171 & -1.7300 \\ -10.0798 & -2.4415 & 7.3354 & 10.4833 & 4.1125 & -7.2516 \\ -1.0474 & -2.5316 & -1.9511 & 0.1222 & 1.8405 & 1.5060 \end{bmatrix}$$

Next, we get the Bernoulli wavelet coefficients,

$$Y = \begin{bmatrix} 0.5931 & 0.1433 & -0.0721 & 0.0858 & -0.0064 & 0.0464 \end{bmatrix}$$

with the help of Bernoulli wavelet coefficients, we get the approximate solution as shown in table 3 and the error analysis is shown in table 4.

Table 3: Numerical result of the example 4.2.

T	Exact	Bernoulli Wavelet (k = 1, M = 6)	Absolute Error
0.0833	0.3902	0.3828	7.45e-03
0.2500	0.5392	0.5378	1.42e-03
0.4167	0.6088	0.6077	1.12e-03
0.5833	0.6527	0.6521	6.21e-04
0.7500	0.6840	0.6834	6.20e-04
0.9167	0.7079	0.7079	5.49e-05

Table 4: Maximum error analysis of the example 4.2

$N = 2^{k-1}M$	Bernoulli Wavelet
k = 1, M = 3	1.6e-02
k = 1, M = 5	9.2e-03
k = 1, M = 6	7.5e-03

Example 3. Let us consider the Abel's integral equation of first kind [16],

$$\frac{2}{105}\sqrt{t}(105 - 56t^2 + 48t^3) = \int_0^t \frac{y(s)}{\sqrt{t-s}} ds. \tag{4.3}$$

Firstly, consider

$$y(t) = Y^T \Psi(t) \tag{4.4}$$

substituting $y(t)$ in Eq. (4.3), we get

$$\frac{2}{105}\sqrt{t}(105 - 56t^2 + 48t^3) = \int_0^t \frac{Y^T \Psi(s)}{\sqrt{t-s}} ds. \tag{4.5}$$

Next, we collocate the point t_i and substitute in Eq. (4.5).

$$\frac{2}{105}\sqrt{t_i}(105 - 56t_i^{\,2} + 48t_i^{\,3}) = \int_0^{t_i} \frac{Y^T \Psi(s)}{\sqrt{t_i - s}} ds. \tag{4.6}$$

Now, we get the system of algebraic equations with unknown coefficients for k = 1 and M = 5 as given,

$$f = \begin{bmatrix} 0.6294 & 1.0564 & 1.3065 & 1.4984 & 1.7100 \end{bmatrix}$$

$$K=\begin{bmatrix} 0.6325 & 1.0954 & 1.4142 & 1.6733 & 1.8974 \\ -0.9494 & -1.1384 & -0.8165 & -0.1932 & 0.6573 \\ 0.8938 & 0.2156 & -0.6325 & -0.8681 & -0.0339 \\ 0.4727 & 1.2808 & 0.9759 & -0.0905 & -0.8107 \\ -4.0997 & -1.6545 & 3.5277 & 4.4946 & -0.5290 \end{bmatrix}$$

By solving this system of equations, we get the Bernoulli wavelet coefficients

$$Y=\begin{bmatrix} 0.9167 & 0 & 0.0373 & 0.0345 & 0 \end{bmatrix}$$

and then substituting these coefficients in Eq. (4.4), we get the accurate solution of Eq. (4.3) with exact solution $y(t)=t^3-t^2+1$ is shown in table 5 and the maximum error is 1.33e-15. Error analysis is shown in figure 1.

Table 5: Numerical results of the example 3.

t	Exact solution	Bernoulli wavelet (k = 1, M = 3)	Bernoulli wavelet (k = 1, M = 5)
0.1	0.991000000000000	0.990793650793651	0.990999999999999
0.2	0.968000000000000	0.955555555555555	0.968000000000000
0.3	0.937000000000000	0.926031746031746	0.937000000000000
0.4	0.904000000000000	0.902222222222222	0.904000000000001
0.5	0.875000000000000	0.884126984126984	0.875000000000001
0.6	0.856000000000000	0.871746031746032	0.856000000000001
0.7	0.853000000000000	0.865079365079365	0.853000000000000
0.8	0.872000000000000	0.864126984126985	0.872000000000000
0.9	0.919000000000000	0.868888888888889	0.918999999999999

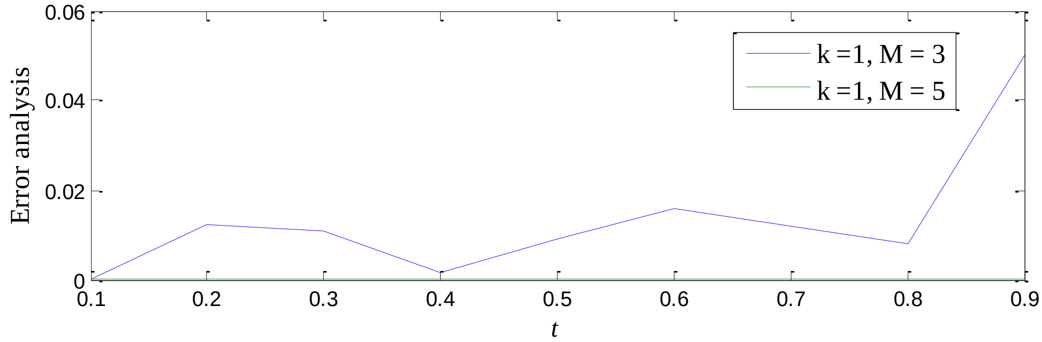

Fig. 1: Error analysis of the example 3.

Example 4. Next, consider [16],

$$t=\int_0^t \frac{y(s)}{\sqrt{t-s}}ds. \tag{4.7}$$

Applying the proposed method, we obtain the approximate solution $y(t)$ of Eq. (4.7) with the help of Bernoulli wavelet coefficients. Numerical solution is compared with exact solution $y(t)=\frac{2}{\pi}\sqrt{t}$ and existing methods is shown in table 6 and figure 2. Error analysis is shown in table 7 and figure 3 is compared with the existing method.

Table 6: Numerical results of the example 4.

t	Exact solution	Bernoulli wavelet (k = 1, M = 6)	Method [16]
0.1	0.201317	0.197156	0.200128
0.2	0.284705	0.284589	0.286092
0.3	0.348691	0.349102	0.347394
0.4	0.402634	0.402358	0.404161
0.5	0.450158	0.449889	0.449568
0.6	0.493124	0.493340	0.492704

0.7	0.532634	0.532707	0.532315
0.8	0.569410	0.568574	0.569156
0.9	0.603951	0.604356	0.603742

Table 7: Error analysis of the example 4.

t	Bernoulli wavelet k = 1, M = 6	Method [16]
0.1	4.16e-03	1.20e-03
0.2	1.15e-04	1.40e-03
0.3	4.11e-04	1.30e-03
0.4	2.75e-04	1.50e-03
0.5	2.68e-04	5.90e-04
0.6	2.17e-04	4.20e-04
0.7	7.30e-05	3.19e-04
0.8	8.35e-04	2.54e-04
0.9	4.05e-04	2.09e-04

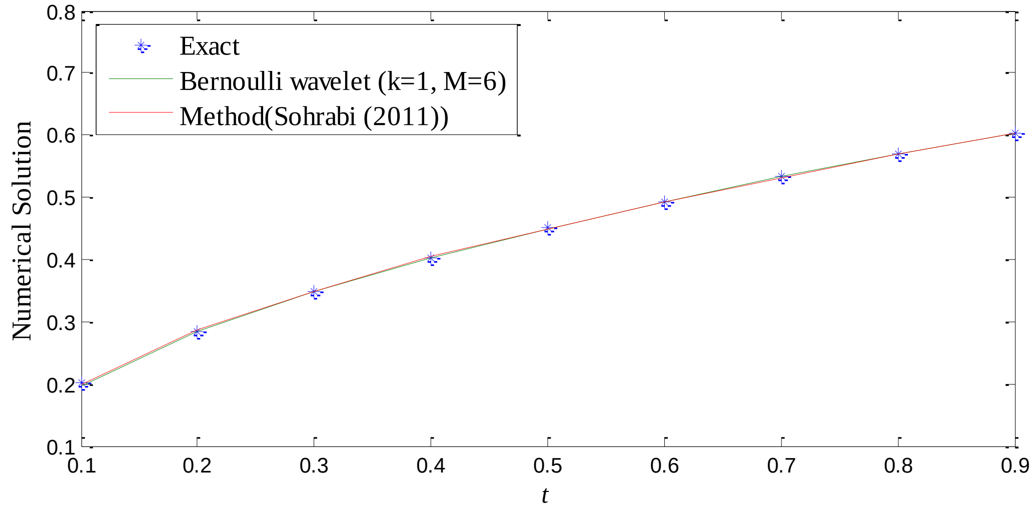

Fig. 2: Comparison of numerical results of the example 4.

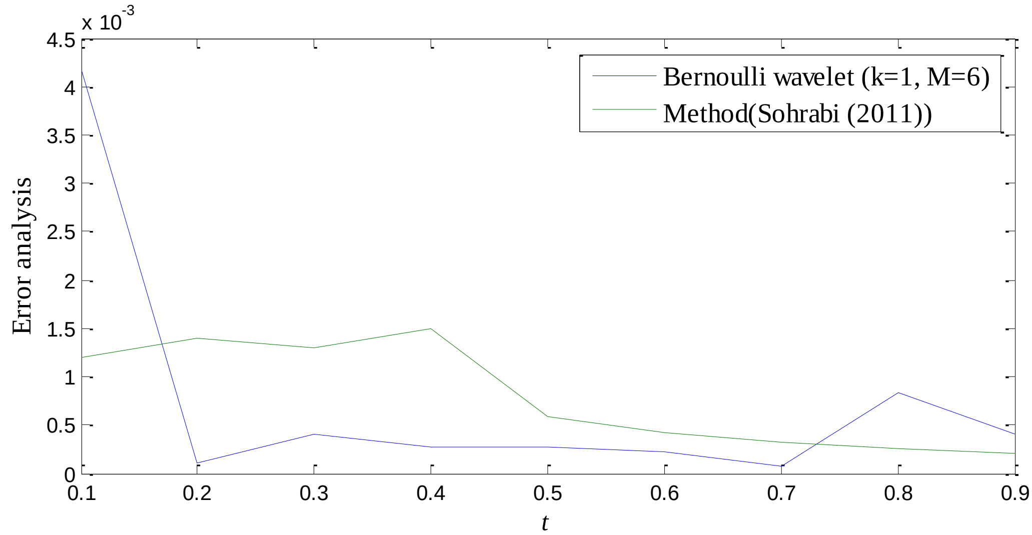

Fig. 3: Comparison of error analysis of the example 4.

5. Conclusion

In this paper, we introduced the Bernoulli wavelet method for the numerical solution of Abel’s integral equations. Using Bernoulli wavelet reduces an integral equation into a system of algebraic equations. Numerical results are highly accuracy with exact ones and existing method [16]. Error analysis shows the accuracy gives better, with increasing the level of resolution N, for better accuracy, and then the larger N is recommended. Hence the present scheme is very easy, accurate and effective.

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