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Brief Study of "Group"

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Group: - the term group was coined by Galois around 1830 to describe sets of one–to-one functions on finite sets that could be grouped together to form a closed set. As is the case with most fundamental concepts in mathematics, the modern definition of a group that follows is the result of a long evolutionary process.

Definition:- Let G be a non empty set together with a binary operation that assigns to each ordered pair (a,b) of elements of G an element in G denoted by ab. G is a group under this operation if the following four properties are satisfied.

1. Closure property:- $\forall\, a \in G, \forall\, b\ \in G$ s.t. $ab\ \in G$
2. Associativity : - The operation associative i.e. (ab)c=a(bc) for all a,b,c in G.
3. Identity: - There is an element e in G s.t. ae=ea=a for all a in G.
4. Inverses:- for each element a in G, there is an element a in G s.t. ab=ba=e

A group is set together with an associative operation s.t. there is an identity , every element has inverse and any pair of element can be combined without going outside the set.

In a group has the property that ab=ba for every pair of elements a and b, then the group is abelian group. A group is non abelian group if there is some pair of elements a and b for which $ab \neq ba$

For example

1. The set of integers Z is a group under ordinary addition.
2. Gl_n is a group under multiplication where
   $Gl_n(IF) = \left\{A = [a_{ij}]_{n\times n} \mid |A| \neq 0, a_{ij}\ \in IF\ \right\}$
3. Sl_n is a group under multiplication where
   $Sl_n(IF) = \left\{A = [a_{ij}]_{n\times n} \mid |A| = 1, a_{ij}\ \in IF\ \right\}$
4. U(n) is a group under multiplication modulo n, where
   U (n)= $\{a \in IN \mid 1 \leq a \leq n; \gcd(a,n) = 1\}$
5. K_4 is a group with identity e, where
   $K_4=\{e, a, b, ab \mid a^2 = e, b^2 = e, ab = ba\}$

Order of elements:-

Order of any element a of group is the least +ve integer n s.t. a^n=e. if no such n exist the O(a) =∞

For example

1. Order of element of Z
   Z=$\{0, \pm1, \pm2, \pm3, \pm4 \ldots\ldots\}$ is a group with identity 0 then
   O(0)=1 and $0 \neq a \in Z\ s.t.\ n.a = 0$ is not possible , $n \in IN \Rightarrow O(a) = \infty$

Then Z has element of order 1 and ∞.

Cyclic group:- A group G is cyclic group if there exist element $a \in G$ such that every element of G is generated by a. Then the element a is called the generator of G.

i.e. $G =< a \geq \{a^n; n \in Z\}$

Theorem: - If G is finite group of order n and G has elements of order n then G is cyclic.

Proof: - Let G be a finite group of order n and $a \in G$ such that O(a)=O(G)=n

Now, by closure property of group, a, a^2,a^3, a^4,……a^{n-1},a^n=e are elements of G

a^n=e as O(a)=n.

Now a, a^2, a^3, a^4,……a^{n-1},a^n=e are distinct elements of G as

Let these elements are not distinct elements and let any two elements are same as A^r=a^s

$\Longrightarrow a^r.a^{-s} = e$

$\Longrightarrow a^{r-s} = e \; and \; r - s < n$

But O(a)=n then $a^{r-s} = e$ is not possible.

Then the supposition is wrong.

$\Longrightarrow a^r \neq a^s$

Then a, a^2, a^3, a^4,……a^{n-1},a^n=e are n distinct elements of G and O(G)=n then G contains exactly n distinct elements then every element of G is generated by a.

Then G is cyclic.

Theorem: - If G is cyclic group then G is abelian But Converse need not be true.

Proof: - Let G be cyclic group the there exist element a in G such that every element of G is generated by a.

Let $x \in G$is any element then x=a^n; $n \in Z$

and $y \in G$ is any element of G then y=a^n; $n \in Z$

such that $x.y = a^n a^m = a^{n+m} = a^{m+n}$ [m+n=n+m as $m, n \in Z$ and Z is abelian group]

$= a^m . a^n$

= y.x

$\Rightarrow x.y = y.x, \forall x, y \in G$

Then G is a abelian group. But Converse need not be true.

For example $K_4=\{e, a, b, ab \mid a^2 = e, b^2 = e, ab = ba\}$ is a abelian group but not cyclic group.

Theorem: - If G is cyclic group and a is generator of G then a^{-1} is also generator of G.

Proof: - Let G be cyclic group and $a \in G$ is generator of G then every element of G is generated by a.

Let x is any element of G then $x=a^n$, $n \in z$

$\Rightarrow x^{-1} = (a^n)^{-1}$[By taking inverse on both side]

$\Rightarrow y = x^{-1} = (a^{-1})^n, n \in Z$

Then y is generated by a^{-1} and y is arbitrary element of G

Then every element of G is generated by a^{-1}.

i.e. $g=\langle a^{-1}\rangle$

then a^{-1} is also a generator of G.

for example: - $1 \in z$ is generator of z then $1^{-1} = -1$ is also generator of Z.

Sub group: - If a Subset H of a group G is itself a group under the operation of g, then H is Subgroup of G.

Subgroup Test: - Let G be a group and H is non empty subset of G. Then H is a sub group of G if ab^{-1} is in H whenever a and b are in H.

Proof: - Since the operation of H is same as that of G and H is non empty subset of G then this operation is associative. Since H is non empty, So Let $x \in H$.

Letting a=x and b=x in hypothesis we have $e=x\,x^{-1}=ab^{-1}$ is in H. Therefore $e \in H$. Now to verify x^{-1} is in H, whenever x is in H. Choose a=e and b=x in the statement of the theorem. Then $ab^{-1}=ex^{-1}=x^{-1}$is in H.

Finally, the proof will be complete when we show that H is closed; that is, if x,y belongs to H , we must show that xy is in H. As y belongs to H the y^{-1} also belongs to H.

So Letting a=x and b=y$^{-1,}$ we have $xy = x(y^{-1})^{-1} = ab^{-1}$ is in H.

For Example: -

H=mz is a Subgroup of Z, $m \in Z$

$H = mZ = \{m.a; a \in Z\}$

Let $x \in H$, then x=ma, where $a \in Z$

And $y \in H$, then y=mb, where $b \in Z$

Such that

$$xy^{-1} = x - y = ma - mb = m(a - b)$$
$$mc \in mZ,\ where\ c = a - b \in z$$
$$\Rightarrow x - y \in mZ$$

Then mZ is subgroup of Z.

Theorem: - Intersection of two subgroup of G is subgroup of G.

Proof: - Let H and K are two subgroup of G.

Now

$H \cap K = \{a|a \in H\ and\ a \in K\}$

As $e \in H\ and\ e \in K$ then

$e \in H \cap K$

$\Rightarrow \emptyset \neq H \cap K \subseteq G$

Let $a \in H \cap K$then $a \in H\ and\ a \in K$

and $b \in H \cap K\ then\ b \in H\ and\ \ b \in K$

As $a \in H$ and $b \in H$ and H is subgroup of G then $ab^{-1} \in H$.

Also. $a \in K$ and $b \in K$ and K is also subgroup of G, then $ab^{-1} \in K$

As $ab^{-1} \in H$ and $ab^{-1} \in K$ then

$ab^{-1} \in H \cap K$

As

$a \in H \cap K \text{ and } b \in H \cap K$

$\Rightarrow ab^{-1} \in H \cap K$

$\Rightarrow H \cap K$ is subgroup of G.

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