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ERROR ESTIMATION FORMULA FOR MODIFIED BETA OPERATORS

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ABSTRACT

In the recent years several researchers proposed and studied beta operators in approximation theory. In the present paper, we establish an error estimation formula for modified beta operators in linear simultaneous approximation. To prove our result, we have used the technique of linear approximating method, namely, Steklov mean .

***Key Words and Phrases*: Simultaneous approximation, Linear combinations, Linear positive operators, Steklov mean, Integral modulus of smoothness.**

Mathematical Subject Classification: **41A25, 41A36.**

1. Introduction

Motivated by the integral modification of Bernstein polynomials by Durrmeyer [3], several researchers have proposed and studied the different family of mixed summation -integral type operators [1, 4, 6, 7, 10, 11, 15, 16]. In 2006, Gupta [8] studied error estimation formula for summation integral type operators. In the present paper we study an error estimation formula in simultaneous approximation for the linear combinations of the operators introduced by Gupta et al. [9]. The modified beta operators introduced by Gupta et al. [9] are defined as

$$B_n(f,x)=\int_0^\infty W_n(x,t)f(t)dt \qquad , \qquad x\in[0,\infty)$$

(1.1)

where

$$W_n(x,t)=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)b_{n,v}(t) \qquad , \qquad b_{n,v}(t)=\frac{1}{\beta(v,n+1)}t^{v-1}(1+t)^{-n-v-1}$$

and $\beta(v,n+1)=(v-1)!n!/(n+v)!$ the Beta function.

It is easily checked that the operators defined by (1.1) are linear positive operators and it is obvious that $B_n(1,x)=1$.Also it is observed that the order of approximation by operators (1.1) is, at best O(n^{-1}), howsoever smooth the function may be. Thus, to improve the order of approximation we may consider some combinations of the operators (1.1).One approach to improve the order of approximation is the iterative combinations due to Micchelli [13], who improved the order of approximation of Bernstein polynomials. However, we cannot apply this approach to the operators (1.1) because for these operators (1.1), we not have $B_n(t-x,x)=0$, which is essential property for making iterative combinations. Yet another approach for improving the order of approximation is the technique of linear combinations which was first considered by May [12] to improve the order of approximation for exponential type operators. In the present paper, we use the later approach, which described as:

Let d_0, d_1, d_2… d_k be (k+1) arbitrary but fixed distinct positive integers. Then the linear combination $B_n(f,k,x)$ of $B_{d_jn}(f,x)$, j = 0, 1, 2…n is defined as

$$B_n(f,k,x)=\frac{1}{\Delta}\begin{vmatrix} B_{d_0 n}(f,x) & d_0^{-1} & d_0^{-2} & \ldots\ldots\ldots\ldots & d_0^{-k} \\ B_{d_1 n}(f,x) & d_1^{-1} & d_1^{-2} & \ldots\ldots\ldots\ldots & d_1^{-k} \\ \ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots\ldots\ldots\ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ B_{d_k n}(f,x) & d_k^{-1} & d_k^{-2} & \ldots\ldots\ldots\ldots & d_k^{-k} \end{vmatrix}$$

(1.2)

where $$\Delta=\begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \ldots\ldots\ldots\ldots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \ldots\ldots\ldots\ldots & d_1^{-k} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 1 & d_k^{-1} & d_k^{-2} & \ldots\ldots\ldots\ldots & d_k^{-k} \end{vmatrix}$$

The above expression (1.2) after simplification may be written as

$$B_n(f,k,x)=\sum_{j=0}^{k} C(j,k)\,B_{d_j n}(f,x)$$

(1.3)

where $C(j,k)=\prod_{\substack{i=0\\ i\neq j}}^{k}\frac{d_j}{d_j-d_i}$, $k\neq 0$ and $C(0,0)=1$.

Some basic properties of $b_{n,v}(x)$ are as follows

(i). $\int_0^\infty t^2\, b_{n,v}(t)\;dt=\frac{v(v+1)}{n(n-1)}$

(1.4)

(ii). $\sum_{v=1}^{\infty} b_{n,v}(x)=(n+1)$ (1.5)

(iii). $\sum_{v=1}^{\infty} v\,b_{n,v}(x)=(n+1)[1+(n+2)x]$

(1.6)

(iv). $\sum_{v=1}^{\infty} v^2\, b_{n,v}(x)=(n+1)[1+3(n+2)x+(n+2)(n+3)x^2]$

(1.7)

(v). $x(1+x)b'_{n,v}(x)=[(v-1)-(n+2)x]b_{n,v}(x)$

(1.8)

where $n\in N$ and $x\in[0,\infty)$.

Throughout this paper, we may assume that $0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty$ and $I_i = [a_i, b_i]$ where *i* =1, 2, 3.

Let $H[0,\infty)$ be the class of all measurable functions defined on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{|f(t)|}{(1+t)^{n+1}} dt < \infty \qquad \text{for some positive integer } n.$$

Obviously the class $H[0,\infty)$ is bigger than the class of all lebesgue integrable functions on $[0,\infty)$. Therefore the operators (1.1) may be applicable for studying a larger class.

The main object of the present paper is to establish an error estimation formula in terms of modulus of continuity [2, 5] in simultaneous approximation for linear combinations of the operators (1.1).

2. PRELIMINARY RESULTS

To prove our main results, we shall require the following auxiliary results:

Lemma 2.1. For $m \in N^0$ (the set of non-negative integers) and *n>m*, let the function $\mu_{n,m}(x)$ be defined as

$$\mu_{n,m}(x) = \left(B_n(t-x)^m, x\right) = \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^\infty b_{n,v}(t)(t-x)^m dt .$$

Then $\mu_{n,0}(x) = 1, \quad \mu_{n,1}(x) = \dfrac{2x+1}{n}$, and there holds the recurrence relation

$$(n-m)\,\mu_{n,m+1}(x) = x(1+x)[\mu'_{n,m}(x) + 2m\mu_{n,m-1}(x)] + (m+1)(1+2x)\mu_{n,m}(x) .$$

Moreover, we have the following consequences about $\mu_{n,m}(x)$:

(i). $\mu_{n,m}(x)$ is a polynomial in *x* of degree *m*,

(ii). for every $x \in [0,\infty)$, $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$

where [α] denotes the integral part of α.

Consequently, on using Holder's inequality, we have from this recurrence relation that

$$B_n\left(|t-x|^r, x\right) = O\left(n^{-r/2}\right) \text{ for each } r>0 \text{ and for every fixed } x \in [0,\infty) .$$

Proof. Since $\mu_{n,m}(x) = B_n\left((t-x)^m, x\right)$, therefore, using linearity property, we have

$$\mu_{n,0}(x) = B_n\left((t-x)^0, x\right) = B_n(1,x) = 1 \qquad \text{and}$$

$$\mu_{n,1}(x) = B_n\left((t-x), x\right) = B_n(t,x) - xB_n(1,x) = \frac{1+2x}{n}$$

To prove the recurrence relation we shall make the use of the following identity

$$x(1+x)b'_{n,v}(x) = [(v-1)-(n+2)x]b_{n,v}(x)$$

Now, we have

$$x(1+x)\mu'_{n,m}(x) = \frac{1}{(n+1)} \sum_{v=1}^{\infty} x(1+x)b'_{n,v}(x) \int_0^\infty b_{n,v}(t)(t-x)^m dt$$
$$- m\frac{1}{(n+1)} \sum_{v=1}^{\infty} x(1+x)b_{n,v}(x) \int_0^\infty b_{n,v}(t)(t-x)^{m-1} dt$$

or

$$x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)]=\frac{1}{(n+1)}\sum_{v=1}^{\infty}[(v-1)-(n+2)x]b_{n,v}(x)\int_0^{\infty}b_{n,v}(t)(t-x)^m\,dt$$

$$=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}[(v-1)-(n+2)t+(n+2)(t-x)]b_{n,v}(t)(t-x)^m\,dt$$

$$=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}t(1+t)b'_{n,v}(t)(t-x)^m\,dt+(n+2)\mu_{n,m+1}(x)$$

$$=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}[(t-x)^2+(1+2x)(t-x)+x(1+x)]b'_{n,v}(t)(t-x)^m\,dt+(n+2)\mu_{n,m+1}(x)$$

$$=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}b'_{n,v}(t)(t-x)^{m+2}\,dt+\frac{(1+2x)}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}b'_{n,v}(t)(t-x)^{m+1}\,dt$$

$$+\frac{x(1+x)}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}(x)\int_0^{\infty}b'_{n,v}(t)(t-x)^m\,dt+(n+2)\mu_{n,m+1}(x)$$

$$=-(m+2)\mu_{n,m+1}(x)-(m+1)(1+2x)\mu_{n,m}(x)-mx(1+x)\mu_{n,m-1}(x)+(n+2)\mu_{n,m+1}(x)$$

Thus, we get the required recurrence relation

$$(n-m)\mu_{n,m+1}(x)=(m+1)(1+2x)\mu_{n,m}(x)+x(1+x)[\mu'_{n,m}(x)+2m\mu_{n,m-1}(x)]$$

The other consequences follow easily from the above recurrence relation.

Lemma 2.2. For $m\in N$ and sufficiently large $n\in N$, there holds the following recurrence relation

$$B_n\left((t-x)^m,k,x\right)=n^{-(k+1)}\left[Q(m,k,x)+o(1)\right]$$

where $Q(m,k,x)$ is a certain polynomial in *x* of degree *m* and $x\in[0,\infty)$ is arbitrary but fixed.

Proof. Using Lemma 2.1, for sufficiently large *n*, we can write

$$B_n\left((t-x)^m,x\right)=\frac{q_0(x)}{n^{[(m+1)/2]}}+\frac{q_1(x)}{n^{[(m+1)/2]+1}}+\ldots+\frac{q_{[m/2]}(x)}{n^m}+\ldots$$

where $q_i(x)$, $i=0$, 1, 2, 3, ... are certain polynomials in *x* of degree at most *m*.

Now, we have

$$B_n\left((t-x)^m,k,x\right)=\frac{1}{\Delta}\begin{vmatrix}\left(\frac{q_0(x)}{d_0n^{[(m+1)/2]}}+\frac{q_1(x)}{d_0n^{[(m+1)/2]+1}}+\ldots\right) & d_0^{-1} & d_0^{-2} & \ldots & d_0^{-k}\\ \left(\frac{q_0(x)}{d_1n^{[(m+1)/2]}}+\frac{q_1(x)}{d_1n^{[(m+1)/2]+1}}+\ldots\right) & d_1^{-1} & d_1^{-2} & \ldots & d_1^{-k}\\ \ldots\ldots\ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots\ldots\ldots & \ldots & \ldots & \ldots & \ldots\\ \left(\frac{q_0(x)}{d_kn^{[(m+1)/2]}}+\frac{q_1(x)}{d_kn^{[(m+1)/2]+1}}+\ldots\right) & d_k^{-1} & d_k^{-2} & \ldots & d_k^{-k}\end{vmatrix}$$

$= n^{-(k+1)}\left[Q(m,k,x)+o(1) \right]$, for each fixed $x \in [0,\infty)$.

This completes the proof of the Lemma 2.2.

Lemma 2.3. For $m \in N^0$, if the m^{th} order moment for the operators (1.1) be defined as

$$U_{n,m}(x)=\frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\left(\frac{v-1}{n+2}-x\right)^m,$$

then, we have $U_{n,0}(x)=1, U_{n,1}(x)=0$ and there holds the recurrence relation

$$(n+2)U_{n,m+1}(x)=x(1+x)[U'_{n,m}(x)+mU_{n,m-1}(x)].$$

Consequently, for each $x \in [0,\infty)$, we have from this relation that

$$U_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right),$$

where [α] denotes the integral part of α .

Proof. Using the definition of $U_{n,m}(x)$ and basic properties of $b_{n,v}(x)$, we obtain

$U_{n,0}(x)=1$ and $U_{n,1}(x)=0$.

Now, we have

$$x(1+x)U'_{n,m}(x)=\frac{1}{(n+1)}\sum_{v=1}^{\infty} x(1+x)b'_{n,v}(x)\left(\frac{v-1}{n+2}-x\right)^m$$
$$-x(1+x)\frac{1}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)m\left(\frac{v-1}{n+2}-x\right)^{m-1}$$

Thus using basic properties of $b_{n,v}(x)$, we get

$$x(1+x)\left[U'_{n,m}(x)+mU_{n,m-1}(x)\right]=\frac{1}{(n+1)}\sum_{v=1}^{\infty}[(v-1)-(n+2)x]b_{n,v}(x)\left(\frac{v-1}{n+2}-x\right)^m$$

$$=\frac{(n+2)}{(n+1)}\sum_{v=1}^{\infty} b_{n,v}(x)\left(\frac{v-1}{n+2}-x\right)^{m+1}$$

$$=(n+2)U_{n,m+1}(x)$$

This completes the proof of the recurrence relation.
The other consequence follows easily from the recurrence relation.

Lemma 2.4([14]). There exist the polynomials $Q_{i,j,r}(x)$ independent of *n* and *v* such that

$$[x(1+x)]^r D^r b_{n,v}(x)=\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+2)^i[(v-1)-(n+2)x]^j Q_{i,j,r}(x)b_{n,v}(x),$$

where D^r is the r^{th} order differentiation operator.

Lemma 2.5. If *C*(*j*, *k*) , *j* = 0, 1, 2, …, *k* are defined as in (1.3), then we have

$$\sum_{j=0}^{k} C(j,k)d_j^{-m}=\begin{cases}1 & \text{for } m=0\\ 0 & \text{for } m=1,2,\dots,k\end{cases}.$$

Proof. From relations (1.2) and (1.3) we get

$$\sum_{j=0}^{k} C(j,k) d_j^{-m} = \begin{vmatrix} d_0^{-m} & d_0^{-1} & d_0^{-2} & \ldots\ldots\ldots\ldots & d_0^{-k} \\ d_1^{-m} & d_1^{-1} & d_1^{-2} & \ldots\ldots\ldots\ldots & d_1^{-k} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ d_k^{-m} & d_k^{-1} & d_k^{-2} & \ldots\ldots\ldots\ldots & d_k^{-k} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & d_0^{-1} & d_0^{-2} & \ldots\ldots\ldots\ldots & d_0^{-k} \\ 1 & d_1^{-1} & d_1^{-2} & \ldots\ldots\ldots\ldots & d_1^{-k} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 1 & d_k^{-1} & d_k^{-2} & \ldots\ldots\ldots\ldots & d_k^{-k} \end{vmatrix}$$

which implies that

$$\sum_{j=0}^{k} C(j,k) d_j^{-m} = \begin{cases} 1 & \text{for } m = 0 \\ 0 & \text{for } m = 1, 2, \ldots, k \end{cases}.$$

Lemma 2.6. For $\mathrm{p,q} \in \mathrm{N}^0$, we have

$$\frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \,|(v-1)-(n+2)x|^p \int_0^{\infty} b_{n,v}(t) \,|t-x|^q \, dt = O(n^{(p-q)/2})$$

Proof. Using Schwarz inequality for integration and then summation, Lemma 2.1 and Lemma 2.3, we have

$$\frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \,|(v-1)-(n+2)x|^p \int_0^{\infty} b_{n,v}(t) \,|t-x|^q \, dt$$

$$\leq \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \,|(v-1)-(n+2)x|^p \left(\int_0^{\infty} b_{n,v}(t) \, dt \right)^{1/2} \left(\int_0^{\infty} b_{n,v}(t) \,(t-x)^{2q} \, dt \right)^{1/2}$$

$$= \frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \,|(v-1)-(n+2)x|^p \left(\int_0^{\infty} b_{n,v}(t) \,(t-x)^{2q} \, dt \right)^{1/2}$$

$$\leq \left(\frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \,[(v-1)-(n+2)x]^{2p} \right)^{1/2} \left(\frac{1}{(n+1)} \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^{\infty} b_{n,v}(t) \,(t-x)^{2q} \, dt \right)^{1/2}$$

$$= O(n^{p/2}) \, O(n^{-q/2}) = O(n^{(p-q)/2})$$

3. ERROR ESTIMATION FORMULA

Now we begin to prove the main results of this section, namely, error estimation formula.

Theorem: Let $1 \leq p \leq 2k+2$, $r \in N$ and the function $f \in H[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$, satisfying $f(t) = O(t^{\gamma})$ as $t \to \infty$ for some $\gamma > 0$. If $f^{(p+r)}$ exists and is continuous on $(a-\Delta, b+\Delta) \subset (0,\infty)$ where $\Delta > 0$, having the modulus of continuity $\omega_{f^{(p+r)}}(\delta)$ on $(a-\Delta, b+\Delta)$, then for sufficiently large $n \in N$, we have

$$\left\| B_n^{(r)}(f,k,\cdot) - f^{(r)}(\cdot) \right\|_{C[a,b]} \leq \max\left\{ C_1 n^{-p/2} \omega_{f^{(p+r)}}(n^{-1/2}) \,,\; C_2 n^{-(k+1)} \right\},$$

where $C_1 = C_1(k,p,r)$ and $C_2 = C_2(k,p,r,f)$.

Proof. Let $\phi(t)$ be the characteristic function of the interval $(a-\Delta, b+\Delta)$. Since $f^{(p+r)}$ exists, therefore, for $t \in [0,\infty)$ and $x \in [a,b]$, we can write

$$f(t)=\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{\left[f^{(p+r)}(\xi)-f^{(p+r)}(x)\right]}{(p+r)!}(t-x)^{p+r}\phi(t)+[1-\phi(t)]\,F(t,x)$$

where ξ lies between t and x, and

$$F(t,x)=f(t)-\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}(t-x)^i$$

Using linearity of $B_n^{(r)}(\,\cdot\,,k,x)$, relation (1.3) and the above expansion of $f(t)$, we have

$$B_n^{(r)}\big(f(t),k,x\big)-f^{(r)}(x)=\left[\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}B_n^{(r)}\big((t-x)^i,k,x\big)-f^{(r)}(x)\right]$$

$$+\sum_{j=0}^{k}C(j,k)\frac{1}{(d_jn+1)}\sum_{v=1}^{\infty}b_{d_jn,v}^{(r)}(x)\int_0^{\infty}b_{d_jn,v}(t)\frac{\left[f^{(p+r)}(\xi)-f^{(p+r)}(x)\right]}{(p+r)!}(t-x)^{p+r}\phi(t)dt$$

$$+\sum_{j=0}^{k}C(j,k)\frac{1}{(d_jn+1)}\sum_{v=1}^{\infty}b_{d_jn,v}^{(r)}(x)\int_0^{\infty}b_{d_jn,v}(t)[1-\phi(t)]F(t,x)dt$$

$=E_1+E_2+E_3$ (say)

Using Lemma 2.2, we obtain

$$E_1=\left[\sum_{i=0}^{p+r}\frac{f^{(i)}(x)}{i!}B_n^{(r)}\big((t-x)^i,k,x\big)-f^{(r)}(x)\right]$$

$$=O\big(n^{-(k+1)}\big)\qquad\text{uniformly for all } x\in[a,b].$$

In order to estimate E_2, let

$$E_{21}=\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}^{(r)}(x)\int_0^{\infty}b_{n,v}(t)\frac{\left[f^{(p+r)}(\xi)-f^{(p+r)}(x)\right]}{(p+r)!}(t-x)^{p+r}\phi(t)\,dt.$$

Since for every $\delta>0$ we have

$$\left|f^{(p+r)}(\xi)-f^{(p+r)}(x)\right|\le\omega_{f^{(p+r)}}\big(|\xi-x|\big)\le\omega_{f^{(p+r)}}\big(|t-x|\big)\le\left(1+\frac{|t-x|}{\delta}\right)\omega_{f^{(p+r)}}(\delta),$$

therefore, using Lemma 2.4 and the above inequality, for all $\delta>0$ we obtain

$$|E_{21}|\le\sum_{v=1}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge0}}\frac{(n+2)^i}{(n+1)}\left|(v-1)-(n+2)x\right|^j\frac{\left|Q_{i,j,r}(x)\right|}{[x(1+x)]^r}b_{n,v}(x)$$

$$\times\int_0^{\infty}b_{n,v}(t)\frac{\left|f^{(p+r)}(\xi)-f^{(p+r)}(x)\right|}{(p+r)!}|t-x|^{p+r}\phi(t)\,dt$$

$$\le\frac{C(x)}{(p+r)!}\sum_{\substack{2i+j\le r\\ i,j\ge0}}\frac{(n+2)^i}{(n+1)}\sum_{v=1}^{\infty}\left|(v-1)-(n+2)x\right|^j b_{n,v}(x)$$

$$\times\int_0^{\infty}b_{n,v}(t)\left(1+\frac{|t-x|}{\delta}\right)\omega_{f^{(p+r)}}(\delta)|t-x|^{p+r}\,dt$$

$$\leq \frac{C(x)}{(p+r)!}\omega_{f^{(p+r)}}(\delta)\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{(n+2)^i}{(n+1)}\sum_{v=1}^{\infty}|(v-1)-(n+2)x|^j b_{n,v}(x)$$

$$\times \int_0^{\infty} b_{n,v}(t)\left(|t-x|^{p+r}+\frac{|t-x|^{p+r+1}}{\delta}\right)dt,$$

where $$C(x)=\sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{[x(1+x)]^{r}}.$$

Applying Lemma 2.6 and choosing $\delta=n^{-1/2}$, we get

$$|E_{21}|\leq \frac{C(x)}{(p+r)!}\omega_{f^{(p+r)}}(n^{-1/2})\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+2)^i\left[O\left(n^{(j-p-r)/2}\right)+n^{1/2}O\left(n^{(j-p-r-1)/2}\right)\right]$$

$$=\frac{C(x)}{(p+r)!}\omega_{f^{(p+r)}}(n^{-1/2})\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\left[O\left(n^{(2i+j-p-r)/2}\right)+O\left(n^{(2i+j-p-r)/2}\right)\right]\leq C_3 n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2})$$

Consequently, applying Lemma 2.5, we get

$$|E_2|\leq C_4 n^{-p/2}\omega_{f^{(p+r)}}(n^{-1/2}) \quad \text{where} \quad C_4=C_4(k,p,r).$$

Now, since for $t\in[0,\infty)\setminus(a-\Delta,b+\Delta)$, we can always choose a $\delta>0$ such that $|t-x|\geq\delta$ for all $x\in[a,b]$, therefore, using Lemma 2.4 we get

$$|E_{31}|=\left|\frac{1}{(n+1)}\sum_{v=1}^{\infty}b_{n,v}^{(r)}(x)\int_0^{\infty}b_{n,v}(t)[1-\phi(t)]F(t,x)\,dt\right|$$

$$\leq \sum_{v=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{(n+2)^i}{(n+1)}|(v-1)-(n+2)x|^j\frac{|Q_{i,j,r}(x)|}{[x(1+x)]^{r}}b_{n,v}(x)\int_{|t-x|\geq\delta}b_{n,v}(t)\,|F(t,x)|\,dt$$

Let *s* be any integer $\geq \max\{\gamma,\,2k+r+2\}$. Then for $|t-x|\geq\delta$ there exists a positive constant C_5 such that

$$|F(t,x)|\leq C_5|t-x|^s$$

Thus, using above inequality and Lemma 2.6, we obtain

$$|E_{31}|\leq C_6\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{(n+2)^i}{(n+1)}\sum_{v=1}^{\infty}|(v-1)-(n+2)x|^j b_{n,v}(x)\int_{|t-x|\geq\delta}b_{n,v}(t)\,|t-x|^s\,dt$$

$$\leq C_6\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{(n+2)^i}{(n+1)}\sum_{v=1}^{\infty}|(v-1)-(n+2)x|^j b_{n,v}(x)\int_{|t-x|\geq\delta}b_{n,v}(t)\frac{(t-x)^{2q}}{\delta^{2q-s}}\,dt$$

$$\leq C_7\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\frac{(n+2)^i}{(n+1)}\sum_{v=1}^{\infty}|(v-1)-(n+2)x|^j b_{n,v}(x)\int_0^{\infty}b_{n,v}(t)\,|t-x|^{2q}\,dt$$

$$= C_7 \sum_{\substack{2i+j\le r \\ i,j\ge 0}} (n+2)^i O\left(n^{(j-2q)/2}\right) \le C_7\, O\left(n^{(r-2q)/2}\right) = C_7\, O\left(n^{-(k+1)}\right),$$

where q is a positive integer greater than $s/2$.

Consequently, using Lemma 2.5, we obtain

$$|E_3| \le C_8 n^{-(k+1)}, \qquad \text{where} \quad C_8 = C_8(k,p,r,f).$$

Finally, collecting the estimates of E_1, E_2 and E_3, we get the required result.

This completes the proof of the theorem.

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