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RESULTS ON CONGRUENCE RELATION OF A LATTICE

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ABSTRACT

In this paper we defined a relation on a lattice and it is observed that the relation is a congruence relation on a lattice. Mainly we have obtained sufficient conditions for a lattice to be reflexive and transitive

KEYWORDS:

Groundnut seeds; Extraction;

: congruence relation, convex sublattice

1. INTRODUCTION In this paper we defined a relation on a lattice and it is observed that the relation is a congruence relation on a lattice. It is observed in Result1 that If ' θ' is an equivalence relation as well as a parital order relation on X then θ (x) = $\Delta_x=\{(x,x)/x\in X\}$.it is also observed that the relation is reflexive iff $\Delta_x=\{(x,x)/x\in X\}\subseteq\theta$ which is observed in Result2. It is also observed in this paper that the relation θ on a set X is symmetric iff $\theta \subseteq \theta^{-1}$. It is also observed in this paper that if θ is a congruence relation on L then [a] θ is a convex sublattice of L. It is also observed in this paper that A reflexive binary relation θ on a lattice 'L' is a congruence relation iff it satisfies properties. Mainly we have obtained sufficient conditions for a lattice to be reflexive and transitive.

Keywords: congruence relation, convex sublattice

Def 1:- A Relation 'θ' on a lattice 'L' is called an equivalence relation if it satisfy the following.

1. Reflexive:- $a \equiv a(\theta)\ \forall\ a\in L$
2. Symmetric:- if $a\equiv b(\theta)$ $a,b \in L$ then $b\equiv a(\theta)$
3. Transitive:- if $a\equiv b(\theta)$ and $b\equiv c(\theta)$ then if $a\equiv c(\theta)$

An equivalence relation θ on a lattice ‘L’ is called a congruence relation if for any $a_0,a_1,b_0,b_1, a\in L$ with $a_0\equiv a_1$ (θ) and $b_0 \equiv b_1$ (θ)then $a_{0\wedge} b_0 \equiv a_{1\wedge} b_1$ (θ)

Example 1:- 1.Let ‘L’ be a lattice for all a, b∈L define ≡ on ‘L’ by
a≡ b(θ) iff a=b then θ is a congruence relation.
Reflexive:- a ≡a(θ) as a=a. for all a ∈L
Symmetric:- Suppose a≡ b(θ) iff a=b iff b=a imply that b≡ a(θ)
Transitive :- Suppose a≡ b(θ) and b≡ c(θ) then a=b and b=c imply that a=c and hence a≡ c(θ).
Compatibility- Let $a_0\equiv a_1$ (θ) and $b_0\equiv b_1$(θ) then $a_0= a_1$and $b_0= b_1$ then
$a_{0\wedge} b_0= a_{1\wedge} b_1$ (θ) and $a_{0\vee} b_0= a_{1\vee} b_1$ (θ) imply that $a_{0\wedge} b_0\equiv a_{1\wedge} b_1$(θ) and $a_{0\vee} b_0\equiv a_{1\vee} b_1$ (θ)
Example 2:- suppose ‘L’ is a lattice. Define a relation ‘≡’ on ‘L’ by a≡b(θ)iff either a< b then ‘θ’ in a congruence relation on ‘L’
Def 2:- suppose ‘L ’is a lattice and a∈L, then the set of all congruence class of the element ‘a’ is denoted as [ã] θ={x∈L/ x≡a(θ)}
Example 3:- If (a,.) is a group and (H,.) is a subgroup of the group (G,.), define ≡ on H by a≡a b(mod H) iff $b^{-1}a\in H$ ∀ a,b ∈G the relation ≡ on H is a congruence relation.
Def 3:- If ‘X’ is a non- empty set and x∈ X then define a relation Δ on’X’ by Δ_x= {(x,x)/x∈X}.
It is to be observed that when the relation θ and Δ_x coincides which is obtained the same in the following result.
Result1:- If ’ θ’ is an equivalence relation as well as a parital order relation on X thenθ (x) = Δ_x={(x,x)/x∈X}
Proof:- we have [x] θ={y∈X/(x,y) ∈θ} let (x,y) ∈θ iff (y,x) ∈θiff x=y iff (x,x)$\in\Delta$ and hence θ =Δ.
Result2:- A relation θ on a set ‘X’ is reflexive iff Δ_x={(x,x)/x∈X}⊆θ
Proof:- Let ‘θ’ be a reflexive relation on set X then clearly Δ ⊆ θ
Conversely let Δ ⊆ θWe have Δ_x={(x,x)/x∈X}which is a reflexive relation on X;
We have $\theta[x] = \{y\epsilon X/(x,y)\epsilon\theta\}$={ x∈X/(x,x) ∈θ}and hence θis reflexive.
Result3: - A relation θ on a set X is symmetric iff θ ⊆ θ^{-1}
Proof: - Let ‘θ’ be a symmetric relation on a set ‘X’

Now we claim that $\theta \subseteq \theta^{-1}$
let $(x,y) \in \theta$ imply that $(y,x) \in \theta$ as θ is symmetric so that $(y,x) \in \theta^{-1}$
and hence $\theta \subseteq \theta^{-1}$.Conversely let $\theta \subseteq \theta^{-1}$ and let $(x,y) \in \theta$ imply that $(x,y) \in \theta^{-1}$
And hence $(y,x) \in \theta$ so that θ is symmetric.
For any relation θ , $\theta \subseteq \theta^{-1}$iff $\theta = \theta^{-1}$
Result4: - A relation θ on a set 'X' is a transitive relation on X iff $\theta 0 \theta \subseteq \theta$.
Proof: -let θ be a transitive relation on a set X.
let $(x,y) \in \theta$ and $(y,z) \in \theta$ imply that $\Rightarrow (x,z) \in \theta$
But for $(x,y) \in \theta$ and $(y,z) \in \theta$
$\Rightarrow (x,z) \in \theta_0 \theta$ and Hence $\theta_0 \theta \subseteq \theta$.
Conversely let θ' be any relation on 'X' such that $\theta_0 \theta \subseteq \theta$.
Now we claim that θ is transitive relation.
let $(x,z) \in \theta_0 \theta$ imply that there exists $y \in X$ such that $(x,y) \in \theta$ and $(y,z) \in \theta$
and hence is a transitive relation on X
Result5 :- If $\theta$and$\varnothing$ are reflexive relations on x then $\theta \subseteq \theta 0 \varnothing$ and $\varnothing \subseteq \theta 0 \varnothing$
Where $\theta(x)=\{y \in x/(x,y) \in \theta\}$ and $\varnothing(x)=\{z \in x/(x,z) \in \varnothing\}$
Proof:- we have $\theta(x)=\{y \in x/(x,y) \in \theta\}$ and $\varnothing(x)=\{z \in x/(x,z) \in \varnothing\}$
Since θ and $\varnothing$ are reflexive , $(x,y) \in \theta \Rightarrow (x,x) \in \varnothing, (x,x) \in \theta$ so that$(x,x) \in \theta 0 \varnothing$
and hence $\theta \subseteq \theta 0 \varnothing$. Let $(x,x) \in \varnothing \Rightarrow (x,x) \in \theta$ so that $\varnothing \subseteq \theta 0 \varnothing$.
Result 6 :- If θ is a congruence relation on L.Then $[a]\theta$ is a convex sublattice of L.
Proof :- we have $[a]\theta = \{X \in L/(x,a) \in \theta\}$.
Since θ is reflexive , for any $a \in L$, $(a,a) \in \theta$
$\Rightarrow a \in [a]\theta$ and hence $[a]\theta \neq \varnothing$.
Now we show that θ is a sublattice of L.
i.e.. θ is closed under meet and join operations.
Let $x,y \in [a]\theta$, Now we have to show that $x \wedge y$, $x \vee y \in [a]\theta$
Since $x \in [a]\theta \Rightarrow (x,a) \in \theta$ and $(y,a) \in \theta$
$\Rightarrow (x \wedge a, a \wedge a) \in \theta$ so that $x \wedge y \in [a]\theta$
Since $(x,a) \in \theta$ and $(y,a) \in \theta \Rightarrow (x \vee y, a) \in \theta \vee \theta = \theta$
So that $x \vee y \in [a]\theta$ Hence θ is closed under join &meet operation.
Now we have to verify the convex property.
Let $x,y \in [a]\theta$ with $x \leq t \leq y$ $\Rightarrow (x,a) \in \theta$ and $(y,a) \in \theta$
Now we claim that $t \in [a]\theta$ i.e. $(t,a) \in \theta$
Since $(a,x) \in \theta, (t,t) \in \theta \Rightarrow (a \wedge t, x \wedge t) \in \theta$
$\Rightarrow (a \wedge t, x) \in \theta$ as $x \leq t$.....(1)
Since $(y,a) \in \theta, (t,t) \in \theta \Rightarrow (y \wedge t, a \wedge t) \in \theta$ so that $(y, a \wedge t) \in \theta$.....(2)
From (1) and (2) $(a \wedge t, x) \in \theta$ and $(t, a \wedge t) \in \theta$ imply that $(x, a \wedge t) \in \theta$.Since $(a \wedge t, t) \in \theta$ so that $(x,t) \in \theta, (a,x) \in \theta \Rightarrow (a,t) \in \theta$ and hence $t \in [a]\theta$ and $[a]\theta$ is a convex sublattice of L

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Theorm7 :- A reflexive binary relation θ on a lattice 'L' is a congruence relation iff the following three properties hold.

for any $x,y,z,t \in L$,

1) $x \equiv y(\theta)$ iff $x \wedge y \equiv x \vee y\ (\theta)$

2) $x \leq y \leq z$ and $x \equiv y\ (\theta)$, $y \equiv z\ (\theta)$ imply that $x \equiv z(\theta)$.

3)If $x \leq y$ and $x \equiv y\ (\theta)$ then $x \wedge t \equiv y \wedge t\ (\theta)$

Proof :- Let θ be a congruence relation on 'L'.

1) Assume $x \equiv y(\theta) \Rightarrow (x,y) \in \theta$. So that $y \in \theta[x]$ for $x \in \theta[x] \Rightarrow (x \wedge y, x \vee y) \in \theta$ as θ Satisfy Substitution Property imply that $x \wedge y \equiv x \vee y(\theta)$.

Conversely let $x \wedge y \equiv x \vee y(\theta) \Rightarrow (x \wedge y, x \vee y) \in \theta \Rightarrow x \wedge y \in \theta[x \vee y]$ and $x \vee y \in \theta[x \wedge y]$

Since $x \wedge y \leq x \leq x \vee y \Rightarrow x \in \theta[x \wedge y]$

And also $x \wedge y \leq y \leq x \vee y \Rightarrow y \in \theta[x \wedge y]$ and hence $(x,y) \in \theta$ so that $x \equiv y\ (\theta)$.

2) Let $x \leq y \leq z$ and $x \equiv y\ (\theta)$ and $y \equiv z\ (\theta)$ implies that $(x,y) \in \theta$ and $(y,z) \in \theta$ so that $(x,z) \in \theta\, _0\theta \subseteq \theta$ so that $(x,z) \in \theta$ imply that $x \equiv z\ (\theta)$.

3) If $x \leq y$, $x \equiv y\ (\theta)$ then $x \wedge t \equiv y \wedge t(\theta)$ and $x \vee t \equiv y \vee t(\theta)$

As $x \equiv y\ (\theta)$,$(x,y) \in \theta$, $t \equiv t\ (\theta) \Rightarrow (t,t) \in \theta$ so that $(x \wedge t, y \wedge t) \in (\theta)$imply that $x \wedge t \equiv y \wedge t(\theta)$ and$(x \vee t, y \vee t) \in (\theta)$ imply that $x \vee t \equiv y \vee t(\theta)$

Conversely let θ be any relation on L satisfy the three conditions of hypothesis θ is a congruence relation on L.

1) Clearly $(x,x) \in \theta\ \forall\ x \in L$ as $x \wedge x \equiv x \vee x\ (\theta)$

2) For $(x,y) \in \theta \Rightarrow x \equiv y$ so that $x \wedge y \equiv x \vee y(\theta)$ and $y \wedge x \equiv y \vee x(\theta)$ imply that $y \equiv x(\theta)$ and hence $(y,x) \in \theta$imply that θ is symmetric.

3)Let $(x,y) \in \theta$ and $(y,z) \in \theta$ then it is easy to observe that $(x,z) \in \theta$ imply that θ is transitive.

4) For $(x,y) \in \theta$ and for any $t \in L$ $(x \wedge t, y \wedge t) \in (\theta)$ and $(x \vee t, y \vee t) \in (\theta)$ imply that θ satisfy substitution property and hence θ is a congruence relation on L.

References:

General Lattice theory by Gabor SAZAZ Clarendron press, Newyork.